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Countable models of small dependent theories

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CONTENTS

NORMATIVE REFERENCES

In the present thesis the following references for standards were used:

SOSE RK 5.04.034-2011. State obligatory standard of education. Postgraduate education. Doctoral studies.

State standard 7.32-2001 (changes dated 2006). Report on scientific-research work. Structure and rules of presentation.

State Standard 7.1-2003. Bibliographic record. Bibliographic description. General requirements and rules.

DEFINITIONS

In the dissertation the following notations with the corresponding definitions are used. Here we present only the basic model-theoretic definitions [1, 2, 3, 4]. For convenience of narration more complex notions will be introduced later.

A **language** or a **signature** $\mathfrak{L}$ (or Σ) consists of the following symbols:

1) for every $f_i \in \mathfrak{F}$ a set of functional symbols $\mathfrak{F}$ and corresponding positive integers n_f;

2) for every $R_j \in \mathfrak{R}$ a set of relational symbols $\mathfrak{R}$ and corresponding positive integers n_R;

3) a set of symbols $\mathfrak{C}$ for the constants.

An $\mathfrak{L}$-**structure** $\mathfrak{M}$ is given by the following:

1) a **universe** (or a universal set) of the structure $\mathfrak{M}$, a non-empty set M;

2) for every functional symbol $f \in \mathfrak{F}$ a function $f^{\mathfrak{M}}: M^{n_f} \to M$;

3) for every relation symbol $R \in \mathfrak{R}$ a set $R^{\mathfrak{M}} \subseteq M^{n_R}$; and

4) for every constant symbol $c \in \mathfrak{C}$ an element $c^{\mathfrak{M}} \in M$ from the universe of the structure.

The **interpretations** $f^{\mathfrak{M}}$, $R^{\mathfrak{M}}$ and $c^{\mathfrak{M}}$, when no ambiguity appears, will be denoted as the symbols f, R and c themselves. We will write an $\mathfrak{L}$-structure as $\mathfrak{M} = \langle M; f^{\mathfrak{M}}, R^{\mathfrak{M}}, c^{\mathfrak{M}} \rangle_{f \in \mathfrak{F}, R \in \mathfrak{R}, c \in \mathfrak{C}}$, or, shorter as $\langle M; f, R, c \rangle_{f \in \mathfrak{F}, R \in \mathfrak{R}, c \in \mathfrak{C}}$ or $\langle M; \mathfrak{L} \rangle$.

An $\mathfrak{L}$-**substructure** of an $\mathfrak{L}$-structure $\mathfrak{N}$, is an $\mathfrak{L}$-structure $\mathfrak{M}$, such that $M \subseteq N$ and the next conditions hold:

1) for all constant symbols $c \in \mathfrak{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{N}}$;

2) for all n-ary function symbols $f \in \mathfrak{L}$, for all $\bar{a} \in M^n$, $f^{\mathfrak{M}}(\bar{a}) = f^{\mathfrak{N}}(\bar{a}) \in M$;

3) for all n-ary relational symbols $R \in \mathfrak{L}$, $R^{\mathfrak{M}} = R^{\mathfrak{N}} \cap M^n$.

A **homomorphism** from a structure $\mathfrak{M}$ to a structure $\mathfrak{N}$ is a mapping $h: M \to N$ that satisfies the next conditions:

1) for every constant symbol $c \in \mathfrak{L}$ of the signature, $h(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$;

2) for every n-ary function symbol $f \in \mathfrak{L}$ and for every $\bar{a} \in M^n$, $h(f^{\mathfrak{M}}(\bar{a})) = f^{\mathfrak{N}}(h(\bar{a}))$; and

3) for every n-ary relational symbol $R \in \mathfrak{L}$ of the signature, and every tuple $\bar{a} \in M^n$, if $\bar{a} \in R^{\mathfrak{M}}$, then $h(\bar{a}) \in R^{\mathfrak{N}}$

An **embedding** is a homomorphism $h: \mathfrak{M} \to \mathfrak{N}$ for which for any n-ary relational symbol R of $\mathfrak{L}$ and for every tuple $\bar{a} \in M^n$ with $\bar{a} \in M^n$, $\bar{a} \in R^{\mathfrak{N}}$ if and only if $\bar{a} \in R^{\mathfrak{M}}$.

An **isomorphism** is a surjective embedding between two structures $\mathfrak{M}$ and $\mathfrak{N}$.

An **automorphism** is an isomorphism from the structure $\mathfrak{M}$ onto itself.

Isomorphic structures are structures $\mathfrak{M}$ and $\mathfrak{N}$, such that there exists an isomorphism function from $\mathfrak{M}$ to $\mathfrak{N}$. It is denoted as $\mathfrak{M} \cong \mathfrak{N}$.

A **term** of a language $\mathfrak{L}$ can be defined inductively by the next rules:

1) each variable is a term;

2) each constant symbol of the language $\mathfrak{L}$ is a term as well;

3) given $f \in \mathfrak{L}$ to be an n-ary function symbol and $t_1, t_1, \ldots, t_n$ to be terms, $f(t_1, t_1, \ldots, t_n)$ is also a term.

An **atomic formula** of the language $\mathfrak{L}$ is an expression of the form $R(t_1, t_1, \ldots, t_n)$, where R is an n-ary relational symbol of the language $\mathfrak{L}$, and $t_1, t_1, \ldots, t_n$ are terms of $\mathfrak{L}$.

A **tuple** $\bar{a} = \langle a_1, \ldots, a_k \rangle \in M^k$ **satisfies** the atomic formula $\varphi(x_1, \ldots, x_k)$ if in the structure $\mathfrak{M}$ if the following holds: $R^{\mathfrak{M}}(t_1(\bar{a}), \ldots, t_k(\bar{a}))$. Satisfaction is denoted in the following way: $\mathfrak{M} \vDash \varphi(a_1, \ldots, a_n)$. In the case when the tuple $\bar{a}$ does not satisfy the formula φ in $\mathfrak{M}$, we denote it as $\mathfrak{M} \nvDash \varphi(a_1, \ldots, a_n)$.

A **formula** and **satisfaction of a formula** in an $\mathfrak{L}$-structure $\mathfrak{M}$ are given by the next rules:

1) each atomic formula is a formula;

2) given a formula $\varphi(x_1, \ldots, x_n)$, the statement $\neg\varphi(x_1, \ldots, x_n)$ is a formula. For $\bar{a} \in M^n$, its negation is satisfiable ($\mathfrak{M} \vDash \neg\varphi(\bar{a})$) if and only if $\mathfrak{M} \nvDash \varphi(\bar{a})$.

3) if $\varphi_1(x_1, \ldots, x_n)$ and $\varphi_2(x_1, \ldots, x_n)$ are formulas, then the statements $(\varphi_1 \wedge \varphi_2)(x_1, \ldots, x_n)$ and $(\varphi_1 \vee \varphi_2)(x_1, \ldots, x_n)$ are formulas as well. Given $\bar{a} \in M^n$, $\mathfrak{M} \vDash (\varphi_1 \wedge \varphi_2)(\bar{a})$ if and only if both $\mathfrak{M} \vDash \varphi_1(\bar{a})$ and $\mathfrak{M} \vDash \varphi_2(\bar{a})$ hold; and $\mathfrak{M} \vDash (\varphi_1 \vee \varphi_2)(\bar{a})$ if and only if $\mathfrak{M} \vDash \varphi_1(\bar{a})$ or $\mathfrak{M} \vDash \varphi_2(\bar{a})$.

4) if $\varphi(x_1, \ldots, x_n)$ is a formula, then $(\exists x_i \varphi(x_1, \ldots, x_n)$ and $\forall x_i \varphi(x_1, \ldots, x_n)$, for $1 \leq i \leq n$ are also formulas. If $\langle a_1, \ldots, a_{n-1} \rangle \in M^{n-1}$, then the formula $\exists x_n \varphi(a_1, \ldots, a_{n-1}, x_n)$ is satisfied in the structure $\mathfrak{M}$ if and only if there is $a_n \in M$ for which $\mathfrak{M} \vDash \varphi(a_1, \ldots, a_{n-1}, a_n)$; and $\mathfrak{M} \vDash \forall x_n \varphi(a_1, \ldots, a_{n-1}, x_n)$ if and only if for each $a_n \in M$ we have $\mathfrak{M} \vDash \varphi(a_1, \ldots, a_{n-1}, a_n)$.

A **sentence** is a formula which has no free variables, that is, every variable occurring in it is in a scope of an existential of universal quantifier.

A **definable subset** of an $\mathfrak{L}$-structure $\mathfrak{M}$ is such a subset D of the set M^n, that there is an element $\overline{b} \in M^m$ and a formula $\varphi(x_1, x_2, \ldots, x_{n+m})$ with

$$D = \{(a_1, \ldots, a_n) \in M^n \mid \mathfrak{M} \vDash \varphi(a_1, \ldots, a_n, b_1, \ldots, b_n)\}.$$

An **elementary substructure** of a structure $\mathfrak{N}$ is a structure $\mathfrak{M}$ (equivalently, $\mathfrak{N}$ is called an elementary extension of the structure $\mathfrak{M}$), denoted as $\mathfrak{M} \prec \mathfrak{N}$, if for every n-formula $\varphi(\bar{x})$ of the language and for all elements $\bar{a} \in M^n$, we have that $\mathfrak{M} \vDash \varphi(\bar{a})$ if and only if $\mathfrak{N} \vDash \varphi(\bar{a})$.

An **elementary embedding** from a structure $\mathfrak{M}$ to a structure $\mathfrak{N}$ is a map f from the universal set M to the universal set N, such that every formula $\varphi(x_1, \ldots, x_n)$ of the language and every $\bar{a} \in M^n$, we have $\mathfrak{M} \vDash \varphi(a_1, \ldots, a_n)$ if and only if $\mathfrak{N} \vDash \varphi(f(a_1), \ldots, f(a_n))$.

Elementary equivalent structures $\mathfrak{M} \equiv \mathfrak{N}$ are $\mathfrak{L}$-structures $\mathfrak{M}$ and $\mathfrak{N}$ such that, for every $\mathfrak{L}$-sentence σ, $\mathfrak{M} \vDash \sigma$ if and only if $\mathfrak{N} \vDash \sigma$. Elementary equivalence of models of $\mathfrak{L}$ is equivalent to having the same theory.

An $\mathfrak{L}$-**theory** is a set of $\mathfrak{L}$-sentences.

A **complete** theory is an $\mathfrak{L}$-theory T such that for every sentence σ of $\mathfrak{L}$ either

5

$\sigma \in T$ or $\neg\sigma \in T$.

A **countable** theory is a theory which has only a countable number of sentences.

A **model** $\mathfrak{M}$ of a theory T is a structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash \varphi$ holds for all sentences $\varphi \in T$.

A **satisfiable** theory is a theory which has a model.

A **sentence** φ **follows** from the theory T if it holds in all models of T, $T \vdash \varphi$.

A **consistent** theory is a theory T such that for every formula φ of the given language, $T \nvdash (\varphi \wedge \neg\varphi)$.

An **inconsistent** theory is a theory which is not consistent.

NOTATIONS AND ABBREVIATIONS

$\mathfrak{L}, \Sigma, ...$	languages
$\mathfrak{M}, \mathfrak{N}, ...$	structures
$M, N, ...$	universes of structures
$a, b, ...$	elements of structures
$\alpha, \beta, ...$	(usually) elements of extensions of structures
$\bar{a}, \bar{b}, ...$	tuples
$p(\bar{x}), q(\bar{y}), ...$	types
$\varphi(\bar{x}), \psi(\bar{y}), ...$	formulas
$\neg$	negation
$\forall, \exists$	existential and universal quantifiers
$\vee, \wedge$	disjunction and conjunction
$\rightarrow, \leftrightarrow$	implication and biconditional
$\Leftrightarrow$	if and only if
$a \in A$	a is an element of the set A
$a \notin A$	a is not an element of the set A
$\subseteq$	subset (substructure) relation
$\subset$	subset (substructure) but not equal
$\cup, \cap, \setminus$	union, intersection, relative complement of sets
$A > B$	all elements of set A are greater than elements of B
$\|A\|$	cardinality of a set A
$\varphi(M) \ (p(M))$	set of realizations
V_p, QV_p	neighbourhood, quasi-neighbourhood
$\vDash, \nvDash$	satisfaction in structure
$\vdash$	entails
$\cong$	isomorphism of structures
$\equiv$	elementary equivalence
$\prec$	elementary substructure
$Th(\mathfrak{M})$	theory of structure $\mathfrak{M}$
$Mod(T)$	class of all models of theory T
$I(T, \lambda)$	number of models of cardinality λ of T
$S_n(T) \ (S_n(A))$	set of all complete n-types (over set A) of theory T
$tp(a/A)$	type of a over the set A
$\perp^a$	relation of almost orthogonality of types
$\perp^w$	relation of weak orthogonality of types
$\mathcal{D}(\mathfrak{M})$	finite diagram (dowry)
$\varphi^c \ (p^c)$	convex closure of a formula (type)
$acl(A) \ (dcl(A))$	algebraic (definable) closure of a set

INTRODUCTION

Actuality of the research theme. At the present time, one of the main tasks of model theory is solving the spectral problem, that is, description for the different classes of theories of the function $I(T, \lambda)$, the function determining the number of non-isomorphic models of a theory T of cardinality λ. One of the insufficiently explored problems is the problem of the description of the number $I(T, \omega)$ of countable non-isomorphic models of the theory T.

Related with this issue is the Vaught's conjecture, or the Vaught's hypothesis, according to which there is no countable theory for which the number of countable models up to an isomorphism is larger than the cardinality of natural numbers and less than the cardinality of real numbers, that is, there exist no theory satisfying the condition $\omega < I(T, \omega) < 2^{\omega}$.

Morley proved [5] that if $I(T, \omega)$ is infinite then it must be ω or 2^{ω} or the cardinality of between ω and the cardinality of continuum. That is, $I(T, \omega) \in \omega \cup \{\omega, \omega_1, 2^{\omega}\}$. Vaught proved [6] that the number of countable nonisomorphic models can not be equal to 2.

A theory is called to be small if the number of all its n-types is no more than countable for any finite n. In the case when a theory is not small it has the maximal number of countable models, that is equal 2^{ω}.

J. Baldwin and A. Lachlan confirmed the Vaught conjecture for the class of uncountably categorical theories [7]. MacKay, Harrington and Shelah in the work [8] confirmed the Vaught conjecture for the class of all omega-stable theories. By using of a theory of orthogonality of 1-types in ordered minimal theories of D. Marker [9], L. Mayer proved the Vaught conjecture for the class of ordered minimal theories [10]. The Vaught conjecture for the class of quite o-minimal theories was confirmed by S.V. Sudoplatov and B.Sh. Kulpeshov [11].

Although the Vaught hypothesis has been proved for various individual classes of theories, in general case the task of counting the number of countable nonisomorphic models is still not solved. One of the classes for which the Vaught conjecture has not been proven yet is the class of dependent theories. Exactly this class is under investigation of this research.

The description of conditions under which complete theories have the maximal, that is $2^{\aleph_0}$, number of countable non-isomorphic models, is an important question in studying the countable spectrum of those theories. For instance, at first, L. Mayer found sufficient conditions for an o-minimal theory to have the maximal number of countable non-isomorphic model; and only after that she moved to proving the Vaught conjecture for o-minimal theories [10, P. 157]. Another example is the work [11, P. 129] by S. Sudoplatov and B.Sh. Kulpeshov, in which the authors indicated the conditions of maximality of countable spectrum, and proved the Vaught conjecture for quite o-minimal theories. In this connection, most of the work will be devoted to finding conditions under which a given theory has the maximal number of countable models up to an isomorphism.

The aims and objectives of the study. The work is devoted to studying countable

spectrum of theories which have a countable number of types. The aims of the work are the following:

1) To find conditions of maximality of a number of countable models.

2) To find a class of dependent theories for which the Vaught conjecture can be solved.

The main provisions for the defense of the dissertation:

1) Given a countable complete theory of (an expansion of) a linear order. If there exists a finite subset of some model of this theory and a non-principal extremely trivial 1-type over this subset, then the given theory has the maximal number, that is 2^ω, of countable non-isomorphic models.

2) If there exists a formula which determines a partial order on tuples of elements such that for every given natural number there exists a finite discrete chain whose length is greater or equal to the number, then the given countable theory has the maximal number of countable models up to an isomorphism.

3) If in a countable complete theory of (an expansion of) a linear order there exists a formula quasi-successor on some non-principal 1-type. Then this theory has 2^ω countable non-isomorphic models.

4) The subclass of the class of dependent theories – the class of weakly o-minimal theories of convexity rank 1 satisfies the Vaught conjecture.

The objects of research are small dependent theories.

The research subjects are countable models of small dependent theories and their number up to an isomorphism.

Research methods include analysis of theories through the use of properties of types. Neighbourhoods in a realization of a type are considered, that is how formulas behave inside the realization set of a given type (for example in [12]), as well as relations of orthogonality between few types are considered: the weak and almost orthogonality between types give an opportunity to understand in which way realizations of these types in models are connected [13]. For example, realization of one type in a model can imply realization of one or more types in the same model, or all realizations of few types can be independent from each other, allowing all possible combinations of realizing-omitting these types in models of the theory. Also, while constructing models, a method based on the Tarski-Vaught test (criterion) is used. This criterion guarantees for a subset of a model that it would be a model of the given theory (and moreover, it would be an elementary submodel of this model).

Novelty of the dissertation research. Problem of description of a countable spectrum of small dependent theories is open at the present time. Classes of theories under the study have not been investigated on a number of countable models.

Theoretical and practical significance of the research. Researches in this area constitute steps in solving the Vaught conjecture. Expected results on the nature of countable models of small dependent theories can be applied to group, ring and field theory.

Connection of the dissertation thesis with the other scientific research works. The dissertation thesis was implemented within the scientific projects of the program of grant financing of fundamental researches in the area of natural sciences of the

Ministry of education and science of the Republic of Kazakhstan "Properties of types in dependent theories" (2015-2017 years, 5125/GF4) and "Conservative extensions, countable ordered models and closure operators" (2018-2020 years, AP05134992).

The work approbation. Results of the work were presented and discussed at the following conferences [14-23] and seminars: Logic Colloquium 2015, University Of Helsinki, Finland, 2015; "Function theory, informatics, differential equations and their applications", Almaty, 2015; "Algebra, analysis, differential equations and their applications", Almaty, 2016; Logic Colloquium 2016, Leeds, United Kingdom, 2016; Annual April scientific conference of the Institute of Mathematics and Mathematical Modeling, Almaty, 2017; International Summer School-Conference "Problems Allied to Universal Algebra and Model Theory", Erlagol-2017, Novosibirsk, Russia, 2017; "Actual problems of pure and applied mathematics", Almaty, 2017; Scientific seminars of the department of algebra and mathematical logic of the Institute of Mathematics and mathematical modeling; Results of this dissertation were discussed with model theory specialists during the scientific training in University of Illinois at Chicago and were presented at Louise Hay Logic Seminar in November 2017.

Publications. Based on results of the dissertation 15 works were published: 5 journal articles (2 in Scopus indexed Journals and 3 in journals recommended by the Committee for Control in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan), and 10 in proceedings of international scientific conferences.

Volume and structure of the dissertation. The work includes the title page, contents, normative references, definitions, notations and abbreviations, introduction, 8 sections, conclusion and references. Total volume of the dissertation is 79 pages, the work contains 1 illustration and 79 literature references.

Main content of the dissertation. The introduction includes actuality of the research theme, aims and objectives, the main provisions for the defense of the dissertation, the research object and subject, methods, novelty and theoretical and practical significance of the research, connection of the dissertation thesis with the other scientific research works, the work approbation, author's publications, and volume, structure and content of the dissertation thesis.

The first section explains the current state of the investigated area of model theory.

The second section gives preliminary information and explains basic tools which will be used throughout the dissertation.

The 3rd section considers dowries (in other words, finite diagrams), meaning, sets of types realized in a given model; and, under the given assumption considers the case of a counterexample of Vaught conjecture.

In the fourth section the notions of weak and almost orthogonality are introduced, some useful properties of types, as well as few theorems connecting orthogonality with the number of countable models are proven.

The 5th section is focused on finding conditions that imply small theories of linear order have the maximum number of countable non-isomorphic models. We introduce different notions of triviality of non-principal types, give examples and prove that a theory of order, which has an extremely trivial type, has $2^{\aleph_0}$ countable models.

In Section 6 countable small theories, which have definable partial order on tuples, are studied, and a theorem on a sufficient condition on such theories to have the maximal number of countable models of is proved.

Section 7 uses approach from the Chapter 6 in order to prove that a theory of a definable linear order which has a so-called formula-quasi-successor has the maximal number of countable non-isomorphic models.

In Section 8 we consider a subclass of dependent theories, namely, the class of weakly o-minimal theories of convexity rank 1. We prove binarity of such theories and show that they satisfy the Vaught conjecture, that is we prove that every weakly o-minimal theory of convexity rank 1 is either countably categorical, Ehrenfeucht, has ω, or 2^ω countable models.

The conclusion lists and generalizes the main results obtained during implementation of the dissertation thesis.

1 HISTORICAL OVERVIEW

Two models of a given language are said to be isomorphic if there exists a bijective function between universes of those models, which preserves basic relations from one structure into the other. It is obvious that in case when two models are isomorphic, then the cardinalities of their universes are equal.

A. Los in [24] conjectured that if a complete theory is categorical in some uncountable cardinality, then it is categorical in all other uncountable cardinalities as well. In 1965 M. Morley [25] confirmed the Los's hypothesis proved homogeneity of all models of categorical theories, while changing the quality of research in model theory, systematically introducing methods of working with types (locally consistent sets of formulas), through introducing ranks of the types and formulas based on study of category of topological spaces of n-types and elementary embeddings. This article, as well as Baldwin-Lachlan's article [7, P. 79], played an important role in development of model theory throughout the next two decades. M. Morley formulated a list of unsolved problems on uncountably categorical theories, which included, besides the above-mentioned question on finite axiomatizability, the question of finiteness of a Morley rank, a question that suggests that the number of countable non-isomorphic models may not be finite. J.T. Baldwin [26] and independently, B.I. Zilber [27], proved finiteness of Morley rank for uncountably categorical theories. T.G. Mustafin and A.D. Taimanov, setting a condition on the Morley tower (that is, an increasing chain of elementary embedded models) given, proved non-finiteness of the number of countable models [28]. The final solution of the problem of Morley about the number of countable models was given in the work by J.T. Baldwin and A. Lachlan [7, P. 79], in which authors proved that an uncountably categorical theory can be either 1 or countable number of countable models. In addition, they reproved M. Morley's theorem, meanwhile establishing that every model of such a theory is characterized by a dimension of a strongly minimal formula. This work defined the nature of researches in model theory, in particular for questions related with counting the number of countable non-isomorphic models, the idea of the dimension began to play a decisive role.

A **spectrum** of a complete theory is a function that assigns to cardinal λ the number of non-isomorphic models of the given theory of cardinality λ, $I(T,\lambda)$.

Main problem. To prove that for every complete theory the spectrum function is non-decreasing for uncountable cardinals.

Saharon Shelah in a series of papers [29-32] proved that for a class of non-stable theories, and stable but theories which are not non-superstable, such a function takes the maximum value on uncountable cardinals. While doing so, he developed the stability theory, now it had become classics in model theory [33]. In addition, it became clear that for the class of totally transcendental theories and the class of superstable but not totally transcendental theories the spectrum functions will be different, and it is necessary to conduct the research of properties of models of these theories by means of rank functions.

Spectrum and rank functions.

Totally transcendental theories.

A theory is called **totally transcendental** if every its type has Morley rank. Every countable ω_1-categorical theory is a totally transcendental theory, the class of totally transcendental theories coincides with the class of ω-stable theories. S. Shelah showed that for every cardinality $\lambda \geq \omega$ there is a saturated model for a given totally transcendental theory. For a totally transcendental theory any two prime over some set models are isomorphic over this set [30, P. 107]. Totally transcendental theories of a rank were investigated by A. Lachlan. He gave a complete description of all possible spectrum functions of rank 2 and degree 1 [34]. B.S. Baizhanov extended the full description of spectrum functions for an arbitrary degree n and rank 2 [35], meanwhile he specified the list of spectrum functions for the degree 1. In year 1978 A. Lachlan introduced an important result in the study of spectral functions, proving that in class of totally transcendental theories, there is no constant functions except for uncountably categorical, and the most important, that every function is non-decreasing.

Theorem [36]. *If T is a totally transcendental theory, then for the spectrum S_T one of the following possibilities holds:*

1) $S_T(\omega_\alpha) = 1$ for every ordinal $\alpha \geq 1$ and $S_T(\omega) \in \{1, \omega\}$;

2) $S_T(\omega_\alpha) = |\alpha|$ for $\alpha \geq \omega$ for all ordinals α and $S_T(\omega_\alpha) = |\alpha + 1|^\omega$;

3) $S_T(\omega_\alpha) = |\alpha + 1|^\omega$ for every $\alpha \geq 1$; and

4) $S_T(\omega_\alpha) \geq \omega^{|\alpha|}$ for every ordinal α.

The case 4) in this classification has great uncertainty. Lachlan hypothesized that in this case spectrums of totally transcendental theory are limited to the following range:

a) $S_T(\omega_\alpha) = \omega^{|\alpha|}$, $\alpha \geq 1$;

b) $S_T(\omega_\alpha) = 2^{\omega_\alpha}$, $\alpha \geq 1$;

c) $S_T(\omega_\alpha) = max(2^\omega, \omega^{|\alpha|})$, $\alpha \geq 1$.

B. Baizhanov [37] extended this list, he constructed for every ordinal $\gamma < \omega_1$ totally transcendental theories which have the following spectrums:

d) $S_T(\omega_\alpha) = min(2^{\omega_\alpha}, \beta(|\alpha + 1|, \gamma))$, $\alpha \geq 1$ (where the cardinal $\beta(\chi, \alpha)$ is defined by induction and is the standard definition in axiomatic set theory);

e) $S_T(\omega_\alpha) = min(2^{\omega,\alpha}, \beta(|\alpha + 1|^\omega, \gamma))$, $\alpha \geq 1$.

About this B.S. Baizhanov's extension it was told in the review article by E.A. Palyutin [38]. A. Lachlan [34, P. 153] and B.S. Baizhanov [39] identified a condition that provides maximality of the number of countable models in all uncountable cardinalities (Lachlan for rank 2, by Baizhanov it was generalized to the class of omega-stable theories), based on dimensions of types associated with various copies of one formula, defined by different constants, connected by a non-trivial relation in the realization of a type (connected type). In the next decade the results of A. Lachlan and B.S. Baizhanov were strengthened, absorbed and blocked by numerous at that time works dedicated to spectrum of superstable and omega-stable theories. For omega-stable theories, the condition of existence of a connected type, magically re-opened in other terms (later named by Shelah, the dimensional order property, dop), together with the condition of an infinite depth constituted a necessary and sufficient condition for the spectrum of omega-stable theories to be maximal.

Spectrum of superstable theories.

Finally, the spectrum problem was solved by Shelah in the second half of the 80's for the class superstable theories, and hence for the class of all complete countable theories [40], and Hart-Hrushovski-Laskowski carefully considered all non-obvious places in the proof of Shelah, closing all gaps in the proof [41]. Note that the list of spectral functions for superstable theories is different from the list for omega-stable theories with adding e'), where instead of ω in the exponent in the definition of beta function e) 2^ω is used.

Number of countable models.

The Lachlan problem. After A. Lachlan proved that every superstable theory can not have a finite number of non-isomorphic models except 1 [42], he formulated the problem that existence of a stable theory which has a finite number of models up to an isomorphism. T.G. Mustafin proved that given a stable theory which has a non-principal superstable type, the countable spectrum of such a theory can not be finite [43]. S.V. Sudoplatov constructed a stable theory that has a finite number of countable models up to an isomorphism [44].

The Vaught Conjecture. The conjecture states that the number of countable non-isomorphic models of a countable theory can be either finite, countable, have cardinality of a continuum, or have an intermediate cardinality between a countable set and the continuum ($I(T, \omega) \in \omega \cup \{\omega, \omega_1, 2^\omega\}$). Vaught proved [6, P. 320], that this number can not be equal to 2. As it was said earlier, a small theory is the theory, number of n-types of which is not maximal for every finite n. If a theory is not small, the number of its countable models is maximal, that is, 2^ω. As mentioned above, J.T. Baldwin and A. Lachlan confirmed the Vaught conjecture for the class of uncountably categorical theories [7, P. 70]. For the class of omega-stable theories the conjecture was confirmed by S. Shelah, L. Harrington and M. Makkai in [8, P. 259]. Laura Mayer, using D. Marker's theory of orthogonality of 1-types in o-minimal theories [9, P. 63], confirmed the Vaught conjecture for the class of o-minimal theories [10, P. 157]. The Vaught conjecture for quite o-minimal theories was proved by S.V. Sudoplatov and B.Sh. Kulpeshov [11, P. 131].

The question about the number of countable models is described in the works of many scientists. The other works on this subject that are referenced by many authors are written by S. Shelah [45], A. Pillay [46] and M. Benda [47]. One more work is the article [48] of S.V. Sudoplatov and R.A Popkov, which classifies the theories which have the continuum number of types (and therefore the maximal number of models) according to different criteria. In his work [49] Enrique Casanovas studied the number of countable models from the different sides of view: semi-isolation, Rudin-Keisler order, smooth classes and closures, predimensions, dimension and stability. In the work [50] B.S. Baizhanov and B. Omarov considered the number of countable nonisomorphic models from the aspect of the notion of finite diagrams. At the present time there is no answer on the Vaught conjecture but model theory specialists continue to work on it, in particular S.V. Sudoplatov [51] jointly with B.S. Baizhanov and V.V. Verbovskiy [52].

The number of countable models of theories with an $\emptyset$-definable relation of a

linear order had been studied in the works [10, P. 146; 11, P. 129; 53-57] and others. The question on the countable spectrum of theories which have a linear and a partial order has a big place in the dissertation, since it is of a big importance in studying the class of dependent, non-stable theories.

2 PRELIMINARY INFORMATION

In this section we introduce basic concepts of model theory as well as the main tools used in this branch of mathematical logic.

2.1 Theories. Basic tools: Tarski-Vaught test, Compactness, Theorem of Existence of a Model, Omitting types

Given an $\mathfrak{L}$-structure $\mathfrak{M}$ we denote $Th(\mathfrak{M}) = \{\varphi \in \mathfrak{L} \mid \mathfrak{M} \vDash \varphi\}$.

Theorem 2.1.1 [3, P. 15] *Let T be a consistent theory. Then the next points are equivalent:*
a) The theory T is complete;
b) All models of T are elementary equivalent;
c) There exists a structure $\mathfrak{M}$ with $Th(\mathfrak{M}) = T$.

Note that if structures are elementary equivalent, then they have the same theory, and the other way, if the structures are of the same theory, then they are elementary equivalent. By $Mod(T)$ we denote the class of all the models of theory T.

Definition 2.1.1 [1, P. 34] *An $\mathfrak{L}$-theory T is called* **inconsistent***, if $T \vdash (\varphi \wedge \neg\varphi)$ for some formula φ. Otherwise the theory it is* **consistent***.*

Theorem 2.1.2 [1, P. 34] *(Gödel's Completeness Theorem) Let T be an $\mathfrak{L}$-theory, φ be a sentence of the language $\mathfrak{L}$. Then $T \vDash \varphi$ if and only if $T \vdash \varphi$.*

Corollary 2.1.1 [1, P. 34] A theory T *is a consistent theory if and only if it is satisfiable.*

An $\mathfrak{L}$-theory T has the **witness property** if given an 1-$\mathfrak{L}$-formula $\varphi(v)$, there is a constant $c \in \mathfrak{L}$ such that $T \vDash (\exists v \ \varphi(v) \rightarrow \varphi(c))$. The theory T is said to be a **maximal theory** if for every sentence φ either φ belongs to the theory T, or its negation [1, P. 34].

Theorem 2.1.3 [1, P. 34]*(Malcev's Compactness Theorem) A theory T is a satisfiable theory if and only if every its finite subset is satisfiable.*

Here are listed some base properties of satisfiable theories.

Lemma 2.1.1 [1, P. 35] *Let we are given is a finitely satisfiable maximal $\mathfrak{L}$-theory T. Then if $\Delta \subseteq T$ is a finite subset and $\Delta \vDash \psi$, then $\psi \in T$.*

Lemma 2.1.2 [1, P. 35] *Let we are given is a finitely satisfiable maximal $\mathfrak{L}$-theory T which has the witness property. Then the theory T has a model. More precisely, if*

k is a cardinal and there are no more than k constant symbols in the language $\mathfrak{L}$, then there is a model $\mathfrak{M} \vDash T$ such that $|\mathfrak{M}| \leq k$.

Lemma 2.1.3 [1, P. 38] *Let T be a finitely satisfiable theory of a language $\mathfrak{L}$, and φ to be an $\mathfrak{L}$-sentence, then either the theory $T \cup \varphi$ or the theory $T \cup \neg \varphi$ is finitely satisfiable.*

Corollary 2.1.2 [1, 38] *Let T be a finitely satisfiable theory of a language $\mathfrak{L}$, then there exists a maximal finitely satisfiable $\mathfrak{L}$-theory T' with $T' \supseteq T$.*

Proposition 2.1.1 [1, P. 40] *Let we are given an $\mathfrak{L}$-theory T which has infinite models. If $\kappa \geq |\mathfrak{L}|$ is an infinite cardinal, then there is a model of T which has the cardinality κ.*

A theory T is called **categorical** if it is a consistent theory and all its models are pairwise isomorphic.

Given an infinite cardinal κ and a theory T which has models of size κ, T is called to be **κ-cathegorical** if any two arbitrary models of cardinality κ of the theory T are isomorphic to each other [1, P. 40]. For an ω-categorical theory we have $I(T, \omega) = 1$.

A structure $\mathfrak{M}$ is said to be **α-categorical** if its theory is α - categorical.

Following the definition, a complete theory which has exactly one countable model up to an isomorphism is called to be **ω-categorical** (or $\aleph_0$-categorical).

Theorem 2.1.4 [1, P. 42] *(Vaught's Test) Given a countable theory T which has no finite models. If the theory T is k-categorical in an infinite cardinal k, then T is complete.*

Proposition 2.1.2 [1, P. 45; 3, P. 18] *(Tarski-Vaught test) Suppose that M is a subset of a universum of a structure $\mathfrak{N}$. Then, M is a universum of an elementary substructure $\mathfrak{M}$ of $\mathfrak{N}$ if and only if for every formula $\varphi(v, \overline{w})$ and every $\overline{a} \in M$, existence of $b \in N$ with $\mathfrak{N} \vDash \varphi(b, \overline{a})$ implies that there is an element c of M such that $\mathfrak{N} \vDash \varphi(c, \overline{a})$.*

The Tarski-Vaught test is one of the main tools used in the dissertation during construction of models.

Theorem 2.1.5 [1, P. 45] *(Upward Löwenheim-Skolem Theorem) Let we are given an infinite $\mathfrak{L}$-structure M, let $\kappa \geq |M| + |\mathfrak{L}|$. Then there exists an $\mathfrak{L}$-structure $\mathfrak{N}$ with $|N| = \kappa$ and $j: \mathfrak{M} \to \mathfrak{N}$ elementary.*

Theorem 2.1.6 [1, P. 46] *(Downward Löwenheim-Skolem Theorem) Let we are given an $\mathfrak{L}$-structure $\mathfrak{M}$ and a subset $X \subseteq M$. Then, there exists an elementary submodel $\mathfrak{N}$ of $\mathfrak{M}$ with $X \subseteq N$ and $|N| \leq |X| + |\mathfrak{L}| + \aleph_0$.*

Definition 2.1.2 [2, P. 42] *Let we are given an $\mathfrak{L}$-structure $\mathfrak{M}$. An* **n-type** *of $\mathfrak{M}$ over a set A is a set $p(\bar{x})$ consisting of formulas of the extended language $\mathfrak{L}(A)$, such that for some $\bar{a}$, a tuple of elements of M, $\mathfrak{M} \vDash \varphi(\bar{a})$ holds for all formulas $\varphi(\bar{x})$ of the type p.*

In this case we say that the model $\mathfrak{M}$ **realizes** the n-type p. If no elements of M realize the type p, $\mathfrak{M}$ is said to **omit** p.

An n-type $p(\bar{x})$ is said to be **complete** if for every formula $\varphi(\bar{x}) \in \mathfrak{M}(A, \bar{x})$ either $\varphi(\bar{x}) \in p(\bar{x})$ or $\neg\varphi(\bar{x}) \in p(\bar{x})$, that is the type is maximal. By $S_n(T)$ we will denote the set of all complete n-types of the theory T. In the dissertation we will usually work with complete types.

A **principal** type is a type $p(\bar{x})$ such that there is a formula $\varphi(\bar{x})$ such that for every $\theta(\bar{x}) \in p(\bar{x})$, $\Gamma \vDash \varphi(\bar{x}) \to \theta(\bar{x})$. This formula φ is said to be an **isolating,** or a **principal** formula of the type p.

A type $p \in S(A)$ is said to be **algebraic**, if there exists an integer $n < \omega$ such that $|p(M)| \leq k$ for every model $\mathfrak{M} \vDash T(A)$. It is easy to see that every algebraic type is isolated.

Proposition 2.1.3 [1, P. 116] *Let we are given an $\mathfrak{L}$-structure $\mathfrak{M}$, let $A \subseteq M$, and p be an n-type over A. Then there is such an elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ that realizes the type p.*

Theorem 2.1.7 [1, P. 125] *(Omitting Types Theorem) Let we are given a theory T of a countable signature $\mathfrak{L}$, and let $p \in S_n(T)$ be a non-isolated n-type. Then the theory T has a model omitting p.*

The previous theorem can be easily generalized to a case with a countable set of nonisolated types.

Theorem 2.1.8 [1, P. 127] *Let we are given a theory T of a countable signature $\mathfrak{L}$. Let p_1, p_2, ...,p_k, ...$\in S_n(T)$ be a countable family of non-principal types of T. Then the theory T has a model omitting every of these n-types.*

Definition 2.1.3 [2, P. 134] *Let $\mathfrak{M}$ be a structure.*
1) *An* **algebraic closure***, $acl(A)$, of the set $A \subseteq M$ is the union of all A-definable finite sets of singletons. That is, $acl(A) = \{b \in M \mid$ there exists a formula $\varphi(x, \bar{a}), \bar{a} \in A,$ and a natural number $n < \omega$ such that $\mathfrak{M} \vDash \varphi(b, \bar{a}) \wedge \exists^{=n}x\varphi(x, \bar{a})\}$.*
2) *A* **definable closure***, $dcl(A)$, of the set $A \subseteq M$ is the union of all A-definable sets of singletons of cardinality 1. That is, $acl(A) = \{b \in M \mid$ there exists a formula $\varphi(x, \bar{a}), \bar{a} \in A,$ such that $\mathfrak{M} \vDash \varphi(b, \bar{a}) \wedge \exists^{=1}x\varphi(x, \bar{a})\}$.*

2.2 Main Kinds of Models and Theories. Dependent Theories

A model $\mathfrak{M}$ of T is called a **prime model** of the theory T if for every other model $\mathfrak{N} \vDash T$ there exists an elementary embedding from $\mathfrak{M}$ into $\mathfrak{N}$. A structure $\mathfrak{M} \vDash T$ is **atomic** if for all tuples $\bar{a} \in M^n$, $tp(\bar{a}/\emptyset)$ is a principal type. For complete countable theories the notions of a prime and of an atomic model are equivalent. In general, a model is prime if and only if it is both prime and atomic. Also, all prime models of the same theory are isomorphic [3, P. 59].

Let k to be an infinite cardinal. A structure $\mathfrak{M}$ of a language $\mathfrak{L}$ is **k-saturated** if for every family $F = \{D_i; i < k\}$ consisting of definable subsets of the structure M with the finite intersection property, there exists an element $a \in M$, with $a \in \bigcap_{i<k} D_i$. A model $\mathfrak{M}$ is called **saturated** if it is $|M|$-saturated, that is for all $A \subset M$, if $|A| < |M|$ and $p \in S_n(A)$, then p is realized in $\mathfrak{M}$.

A weak form of saturation is homogeneity. A model $\mathfrak{M} \vDash T$ is called **homogeneous** if for every subset $A \subseteq M$, with cardinality of A less than the cardinality of M, for every partial elementary map $f: A \to M$ and every $a \in M$, there exists a function $f^* \supseteq f$ for which $f^*: A \cup \{a\} \to M$ is also a partial elementary mapping.

Theorem 2.2.1 [1, P. 138] *If a model is saturated, then it is homogeneous.*

Theorem 2.2.2 [1, P. 145] *Any two countable models of a same complete theory of a countable language which are homogeneous and realize the same $\emptyset$-definable n-types for every $n \geq 1$, are isomorphic.*

The previous theorem implies that if there is a countable family of nonisomorphic models which realize the same $\emptyset$-definable n-types for every $n \geq 1$ (that is, they have the same finite diagram), then all of them, except maybe one model, are not homogeneous.

A model is **universal** if every model of the given theory of the same cardinality can be elementarily embedded into it.

Below we present the main kinds of theories. Examples and a nice visual representation can be found at the website [58].

A formula $\varphi(\bar{x}, \bar{y})$ has the **independence property** (or IP), if there exist two sequences $\bar{a}_i$, $i < \omega$ and $\bar{b}_I$, with $I \subseteq \omega$ for which $\vDash \varphi(\bar{a}_i, \bar{b}_I) \Leftrightarrow i \in I$. A theory is called **dependent** (NIP) [59] if all its formulas are NIP, that is no formula of the theory has the independence property.

A formula $\varphi(\bar{x}, \bar{y})$ has the **strict order property** (in short SOP) [59, P. 33], if there are such tuples $\bar{a}_i$, $i < \omega$, for which

$$\vDash \exists \bar{x}(\varphi(\bar{x}, \bar{a}_j \wedge \neg\varphi(\bar{x}, \bar{a}_i)) \Leftrightarrow i < j.$$

A theory is called to be **NSOP** if no its formula has the strict order property.

A countable theory T is λ-stable, if $|S_n(A)| \le \lambda$ for every subset A with $|A| \le \lambda$ for all structures $\mathfrak{M} \vDash T$. A theory is said to be **stable** [2, P. 308] if there is an infinite cardinal λ such that T is λ-stable. A theory T is called **superstable** if there exists such a cardinal κ that the theory T is λ-stable for every $\lambda > \kappa$ [2, P. 310].

A theory is a **strongly minimal** theory if in every structure $\mathfrak{M} \vDash T$ every definable set of $\mathfrak{M}$ is either finite or cofinite [1, P 78]. Every strongly minimal theory is ω-stable, every ω-stable theory is superstable, every superstable theory is stable, and every stable theory is dependent (NIP) and NSOP.

A class of theories that both NIP and SOP includes an important subclass of o-minimal theories, which, in its turn, a subclass of weakly o-minimal theories.

A theory T of (an extension of) a linear order is called an **o-minimal** theory [60] if in every model $\mathfrak{M} \vDash T$ every definable subset of M can be represented as a finite union of points in M and intervals with endpoints in M. The theory T is said to be **weakly o-minimal** if for each model $\mathfrak{M} \vDash T$ every definable subset of M is a union of a finitely number of convex sets in M. This class will be given more attention in the Section 8.

2.3 Number of Countable Models. Small Theories

Let T be a countable complete theory, $\mathfrak{M}$ be a model of T. The number of different up to an isomorphism models of T of cardinality λ is denoted by $I(T, \lambda)$.

Theories with a finitely many, but more than one, countable models are called **Ehrenfeucht** theories.

Theorem 2.3.1 [61] *Let $\mathfrak{L}$ be a countable first order language. Then there are at most 2^ω many countable models for the language $\mathfrak{L}$.*

A theory which has no more than countable number of no more than countable models is small. But the converse is not true.

Lemma 2.3.1 [49, P. 2] *Let we are given a theory T of a countable language. Then $|S_n(T)| > \omega$ implies $|S_n(T)| = 2^\omega$, $n < \omega$.*

Let us recall that

Definition 2.3.1 [3, P. 53] *A theory T is **small** if for every natural number $n < \omega$, $|S_n(\emptyset)| \le \omega$.*

Lemma 2.3.2 [49, P. 1] *The following points are equivalent:*
1) The theory T is a small theory;
2) For every $n < \omega$, and for all finite sets A, $|S_n(A)| \le \omega$;
3) For all finite sets A, $|S_1(A)| \le \omega$;
4) T has a countable saturated model.

Proposition 2.3.1 [49, P. 2] *1) All the countable ω-cathegorical theories are small theories.*
2) All the ω-stable theories are small theories.

Proposition 2.3.2 [4, P. 154] *If we are given a small theory T, then for every finite subset of a set A there exists a prime model over A.*

Fact 2.3.1 [49, P. 2] *If $k \geq |T|$, then $I(T,k) \leq 2^k$.*

Theorem 2.3.2 [49, P. 2] *If T is a non-small theory, then $I(T,\omega) = 2^\omega$, that is the number of countable models of T up to an isomorphism is maximal.*

Thereby, Theorem 2.3.2 allows us to narrow down the problem of countable spectrum of complete countable theories to investigating the countable spectrum only of those theories, which are small.

3 FINITE DIAGRAMS

Let T be a countable complete theory. As always, by $S(T)$ we will denote the set of all the complete types of the given theory T over an empty set.

Hypothesis 3.1 [50, P. 1] *Let we are given the following countable sets of types $\{p_i \in S(T) \mid i < \omega\}$ and $\{q_i \in S(T) \mid i < \omega\}$, and all the types are non-principal. If for every natural number $n < \omega$ there exists a model $\mathfrak{M}_n$ of theory T, in which for every $i \leq n$ the types p_i are realized and the types q_i are omitted, then there exists $\mathfrak{M} \vDash T$ which is countable and such all the types p_i, $i < \omega$ are realized in $\mathfrak{M}$ and all the types q_i, $i < \omega$ are omitted in this structure.*

Definition 3.1 *Given a structure $\mathfrak{M} \vDash T$ $\mathcal{D}(\mathfrak{M})$ denotes the set consisting of all complete types which are realized in the structure $\mathfrak{M}$: $\mathcal{D}(\mathfrak{M}) = \{p \in S(T) \mid \mathfrak{M} \vDash p\}$. We call the set $\mathcal{D}(\mathfrak{M})$ the **finite diagram** (or **dowry**) of the model $\mathfrak{M}$.*

In [62] the following theorem has been proved.

Theorem 3.1 [62, P. 50] *Under condition of Hypothesis 3.1, if the theory T has more than ω of different finite diagrams, then $I(T, \omega) = 2^\omega$.*

Proof of Theorem 3.1 By Λ let us denote the set of all the finite diagrams of all models of theory T : $\Lambda = \{D \mid \exists \mathfrak{M} \in Mod(T), \mathcal{D}(\mathfrak{M}) = D\}$.

Lemma 3.1 *Let $|S(T)| = \omega$, $|\Lambda| \geq \omega_1$, then there is p_0, a type for which $|\Lambda_0| \geq \omega_1$ and $|\Lambda_1| \geq \omega_1$, where*

$$\Lambda_0 = \{\mathcal{D}(\mathfrak{M}) \mid p_0 \in \mathcal{D}(\mathfrak{M}) \in \Lambda\};$$
$$\Lambda_1 = \{\mathcal{D}(\mathfrak{M}) \mid p_0 \not\in \mathcal{D}(\mathfrak{M}) \in \Lambda\}.$$

Proof of Lemma 3.1 Let $p_1, p_2, \ldots, p_n, \ldots$ be a list of all non-principal types from $S(T)$. For any n a type p_n divides the set Λ into two parts, $\Lambda_0^{(n)}$ and $\Lambda_1^{(n)}$, where

$$\Lambda_0^{(n)} = \{\mathcal{D}(\mathfrak{M}) \mid p_n \in \mathcal{D}(\mathfrak{M}) \in \Lambda\};$$
$$\Lambda_1^{(n)} = \{\mathcal{D}(\mathfrak{M}) \mid p_n \not\in \mathcal{D}(\mathfrak{M}) \in \Lambda\}.$$

Towards a contradiction assume that the lemma is not false. Then, for every natural number n, we have either $|\Lambda_0^{(n)}| \leq \omega$ or $|\Lambda_1^{(n)}| \leq \omega$.

Let $B_n = \Lambda_0^n$, if $|\Lambda_0^{(n)}| \leq \omega$; $B_m = \Lambda_1^n$, if $|\Lambda_1^{(n)}| \leq \omega$.

Since for any n $|B_n| \leq \omega$, $|\bigcup_{n<\omega} B_n| \leq \omega$. And therefore $|\Lambda \setminus \bigcup_{n<\omega} B_n| \geq \omega_1$. Take two different elements D_1 and D_2 from $\Lambda \setminus \bigcup_{n<\omega} B_n$. There exists a type p_m such that $p_m \in D_1$ and $p_m \not\in D_2$.

There may be two cases:

1) $B_m = \Lambda_0^{(m)}$. In this case, $p_m \in D_1 \in \Lambda_0^{(m)} = B_m$. But $D_1 \in \Lambda \backslash \bigcup_{n < \omega} B_n$, therefore, $D_1 \not\in B_m$. And we obtain a contradiction.

2) $B_m = \Lambda_1^{(m)}$. In this case, $p_m \not\in D_2 \in \Lambda_1^{(m)} = B_m$. But $D_2 \in \Lambda \backslash \bigcup_{n < \omega} B_n$, therefore, $D_2 \not\in B_m$. And we obtain a contradiction.

□

In order to proof the Theorem 3.1 consider two cases:

1) $|S(T)| > \omega$. In this case $|S(T)| = 2^\omega$, $I(T, \omega) = 2^\omega$.

2) $|S(T)| = \omega$. Consider an arbitrary listing $t_1, t_2, \ldots, t_n, \ldots$; $t_n \in S(T)$ of all non-principal types from $S(T)$.

We will construct a tree by the following steps:

Step 1. By the Lemma 0.1 we will find the smallest number m, such that

$$\Lambda_0 = \{\mathcal{D}(\mathfrak{M}) \quad | \quad t_m \in \mathcal{D}(\mathfrak{M}) \in \Lambda\};$$
$$\Lambda_1 = \{\mathcal{D}(\mathfrak{M}) \quad | \quad t_m \not\in \mathcal{D}(\mathfrak{M}) \in \Lambda\};$$

$$|\Lambda_0| \geq \omega_1, \ |\Lambda_1| \geq \omega_1.$$

Step k-1. On this stage we will have 2^{k-1} disjoint sets Λ_τ with $|\Lambda_\tau| \geq \omega_1$, where $\tau \in \{0,1\}$ and length of τ is equal to $k - 1$.

Step k. For any τ let m_τ be the smallest with the property

$$\Lambda_{\tau_0} = \{\mathcal{D}(\mathfrak{M}) \quad | \quad t_{m_\tau} \in \mathcal{D}(\mathfrak{M}) \in \Lambda_\tau\};$$
$$\Lambda_{\tau_1} = \{\mathcal{D}(\mathfrak{M}) \quad | \quad t_{m_\tau} \not\in \mathcal{D}(\mathfrak{M}) \in \Lambda_\tau\};$$

$$|\Lambda_{\tau_0}| \geq \omega_1, \ |\Lambda_{\tau_1}| \geq \omega_1.$$

On this step we have 2^k sets, each of which has cardinality greater or equal to ω_1, and for any $\tau_1 \neq \tau_2$, $\Lambda_{\tau_1} \cap \Lambda_{\tau_2} = \emptyset$.

Each branch of 2^ω branches of the obtained tree, will be characterized by a sequence $t_m, t_{m_{\tau_1}}, \ldots, t_{m_{\tau_k}}, \ldots$ of types, which we can divide according to belonging of the type $t_{m_{\tau_n}}$ to the finite diagrams of the set $\Lambda_{\tau_{n+1}}$ into two sequences: $p_0, p_1, \ldots, p_k, \ldots$ and $q_0, q_1, \ldots, q_k, \ldots$. If $t_{m_{\tau_i}} = p_k$, then, beginning from i there are models $\mathfrak{M}_n$ of T, $n > i$, such that p_k is realized in all the models $\mathfrak{M}_n$. If $t_{m_{\tau_i}} = q_k$, then, beginning from i there are models $\mathfrak{M}_n \vDash T$, $n > i$, which omit the type q_k.

Therefore, by the Hypothesis 3.1 there are a countable model $\mathfrak{M}$, which will realize all the p_k and omit all the q_k. And all models corresponding to the different branches of the tree will be non-isomorphic since they differ in the collections of types. Thus, there are 2^ω countable non-isomorphic models.

□

Corollary 3.1 [62, P. 52] *Under condition of Hypothesis 3.1, if there exists a theory T which is countable and has ω_1 countable nonisomorphic models, then there is such a finite diagram $D \in \Lambda$, for which $D = \mathcal{D}(\mathfrak{M}_i)$, $\mathfrak{M}_i \in Mod(T)$, $i < \omega_1$.*

That is, there are at least ω_1 models of T having the same diagram.

Proof of Corollary 3.1 The Theorem 3.1 implies that

$$|\Lambda| > \omega \Rightarrow I(T, \omega) = 2^\omega.$$

Then, $I(T, \omega) < 2^\omega$ implies that $|\Lambda| \leq \omega$. Therefore $|\Lambda| \leq \omega$.

Suppose that the Corollary 0.1 does not holds. Then, for any finite diagram D_i $|\{\mathfrak{A} \mid \mathfrak{A} \vDash T, \mathcal{D}(\mathfrak{A}) = D_i\}| \leq \omega$.

Therefore, $|\bigcup_{D_i \in \Lambda} \{\mathfrak{A} \mid \mathfrak{A} \vDash T, \mathcal{D}(\mathfrak{A}) = D_i\}| \leq \omega$, what is a contradiction with our assumption that $I(T, \omega) = \omega_1$.

□

Corollary 3.2. [15, P. 170] *Under condition of Hypothesis 3.1, if there exists a countable complete theory which has ω_1 countable nonisomorphic models, then there exists a finite diagram which has ω_1 countable nonisomorphic non-homogeneous models.*

4 WEAK AND ALMOST ORTHOGONALITY

Definition 4.1 [45, P. 230] *Two types $p(\bar{x})$ and $q(\bar{y})$ from $S(A)$ are called* **weakly orthogonal**, *if $p(\bar{x}) \cup q(\bar{y})$ has a unique extension to a complete type over the set A.*

The relation of weak orthogonality of two types p and q is denoted by $p \perp^w q$.

The types p and q are **not weakly orthogonal**, which is written as $p \not\perp^w q$, if the number of extensions of $p(\bar{x}) \cup q(\bar{y})$ to a complete type at least equals to 2.

It is obvious, that

Fact 4.1 *If in a countable small theory T for any finite set of types $\{p_1(\bar{x}_1), \ldots, p_n(\bar{x}_n) \mid p_i \in S(T), n < \omega\}$, $(p_1 \cup p_2 \cup \ldots \cup p_n)(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is a complete type, then $I(T, \omega) = 2^\omega$.*

Definition 4.2 [62, P. 43] *The types p and q are called to be* **not almost orthogonal** *if for some formula $\varphi(\bar{x}, \bar{y})$, and some model $\mathfrak{M} \vDash T$ realizing p, such that for a tuple $\bar{\alpha} \in p(M)$, we have $\emptyset \neq \varphi(M, \bar{\alpha}) \subset q(M)$.*

The relation of not almost orthogonality of types p and q is denoted by $p \not\perp^a q$. Otherwise, the types are called **almost orthogonal**, $p \perp^a q$.

The notions of weak and almost orthogonality of types are of a big importance in the direction of the dissertation, since, even when they are not noted explicitly, they are present in every main concept and proof we will encounter.

Definition 4.3 *Let Γ be a locally consistent set of formulas, q be a type from $S(T)$. The family Γ is* **almost orthogonal** *to the type q, written as $\Gamma \perp^a q$, if every extension of the set Γ is almost orthogonal to the type q.*

Proposition 4.1 *The types p and q will be not almost orthogonal, $q \not\perp^a p$ if and only if there exists such a formula $\varphi(\bar{x}, \bar{y})$, that for each model $\mathfrak{M}$ realizing T with the condition $\mathfrak{M} \vDash p$, for every $\bar{\alpha} \in p(M)$ we have $\emptyset \neq \varphi(M, \bar{\alpha}) \subset q(M)$.*

Proof of Proposition 4.1 The part "if" is obvious.

We know that $\exists \bar{\alpha} \; \varphi(M, \bar{\alpha}) \subset q(M) = \bigcap_{\theta \in q} \theta(M)$. Therefore, we have that $\varphi(M, \bar{\alpha}) \subset \theta(M)$ for every formula θ from the type q.

The following holds $\mathfrak{M} \vDash \forall \bar{x}(\varphi(\bar{x}, \bar{\alpha}) \rightarrow \theta(\bar{x}))$ for every θ from q. Denote $\forall \bar{x}(\varphi(\bar{x}, \bar{\alpha}) \rightarrow \theta(\bar{y}))$ by $K_\theta(\bar{\alpha})$. $K_\theta(\bar{\alpha}) \in p$.

Let, for some $\mathfrak{M}' \vDash T$, $\bar{\alpha}' \in p(M')$. Therefore, $\mathfrak{M}' \vDash K_\theta(\bar{\alpha}')$. $\mathfrak{M}' \vDash \forall \bar{x}(\varphi(\bar{x}, \bar{\alpha}') \rightarrow \theta(\bar{x}))$ for every formula θ from the type q.

We have that $\varphi(M', \bar{\alpha}') \subset \theta(M')$ for every formula θ from the type q. Which follows that, $\varphi(M', \bar{\alpha}') \subset \bigcap_{\theta \in q} \theta(M') = q(M')$.

$\square$

The following are important facts about types and their relations which are used during the next sections.

Lemma 4.1 *1) Let p, $q \in S(A)$, p be principal, q be non-principal. Then $p \perp^a q$.*

2) Let we are given a small theory T, and let $\mathfrak{M} = \langle M, \Sigma \rangle$, then for each formula $\psi(\bar{x}, \bar{b})$, $\bar{b} \in M$ there is a subformula $\psi_0(\bar{x}, \bar{b})$ such that $\psi_0(\bar{x}, \bar{b})$ determines an isolated type over $\bar{b}$.

3) [13, P. 47] Let p, $q \in S(A)$, q be non-principal, $\mathfrak{M} = \langle M, \Sigma \rangle$ be a model of a small theory T, and $A \subset M$ be a finite set. Then the types p and q are almost orthogonal if and only if for $\bar{c} \vDash p$ and every type $q' \in S(A\bar{c})$ for which $q(\bar{x}) \subset q'(x, \bar{c})$ we have that the type $q'(\bar{x}, \bar{c})$ is not principal.

4) If $tp(\bar{c}\bar{d}/\bar{b}) \not\perp^a q(x, \bar{b})$ is non-principal, $tp(\bar{d}/\bar{c})$ is principal, then
$tp(\bar{c}/\bar{b}) \not\perp^a q(x, \bar{b})$. Equivalently, if $tp(\bar{c}/\bar{b}) \perp^a q(x, \bar{b})$ is non-principal, $tp(\bar{d}/\bar{c})$ is principal, then $tp(\bar{c}\bar{d}/\bar{b}) \perp^a q(x, \bar{b})$.

Let $q(\bar{x}, \bar{b})$ be non-principal, $tp(\bar{c}\bar{d}/\bar{b}) \perp^a q(x, \bar{b})$ and $tp(\bar{d}/\bar{c}\bar{b})$ be principal, then $tp(\bar{c}/\bar{b}) \perp^a q(x, \bar{b})$. If $tp(\bar{c}/\bar{b}) \perp^a q(x, \bar{b})$ and $tp(\bar{d}/\bar{c}\bar{d})$ is principal, then $tp(\bar{c}\bar{d}/\bar{b}) \perp^a q(\bar{x}, \bar{b})$.

5) Let $\bar{b} \in M$, $\bar{c} \in N \ M$, $\psi_0(x, \bar{b}, \bar{c})$ defines a principal type over $\bar{b}\bar{c}$, $q(x, \bar{y}) \in \mathcal{D}(\mathfrak{M})$. Then the following holds: if $q(N, \bar{b}) \cap M = \emptyset$ then $\psi_0(N, \bar{b}, \bar{c}) \subset q(N, \bar{b})$ or $\psi_0(N, \bar{b}, \bar{c}) \cap q(N, \bar{b}) = \emptyset$.

Proof of Lemma 4.1 1) Let us suppose that $p \not\perp^a q$. Then for some realization $\bar{a} \in p(M)$ there is a formula $\varphi(\bar{x}, \bar{y})$ having the following property $\emptyset \neq \varphi(M, \bar{a}) \subset q(M)$. Since $p(\bar{y})$ is a principal type, there is exists isolating formula $\theta(\bar{y})$ for which $p(M) = \theta(M)$. Now let us consider the following A-formula $H(\bar{x}) := \exists y(\theta(y) \wedge \varphi(\bar{x}, \bar{y}))$. So we have $H(M) \subset q(M)$, what contradicts to q being non-principal.

2) Note that if the formula $\psi(\bar{x}, \bar{b})$ has no subformulas defining an isolating type, then every its subformula has the same property. Consider an arbitrary subformula $\psi_1(\bar{x}, \bar{b}) \subset \psi(\bar{x}, \bar{b})$. Then the formula $\psi_0(\bar{x}, \bar{b}) := \psi(\bar{x}, \bar{b}) \wedge \neg\psi_1(\bar{x}, \bar{b})$ is a proper subformula of $\psi(\bar{x}, \bar{b})$. Therefore for every finite sequence $\tau = \langle \tau_1, \tau_2, \ldots, \tau_n \rangle$ of 0's and 1's we can choose the following sequence of $\bar{b}$-definable formulas: $\psi_\tau(\bar{x}, \bar{b}) \subset \psi_{\tau_1, \tau_2, \ldots, \tau_n}(\bar{x}, \bar{b}) \subset \ldots \subset \psi_{\tau_1}(\bar{x}, \bar{b})$. The last means existence of an infinite 2-branching tree of $\bar{b}$-formulas, what contradicts with T being small.

3) Now let $p \perp^a q$. Towards a contradiction suppose that that is there is $\bar{c}_0 \in M$ with $\bar{c}_0 \vDash p$, there is a type $q' \in S(A\bar{c}_0)$, such that $q'(\bar{x}, \bar{c}_0)$ is principal, and $q'(M, \bar{c}_0) \subset q(M)$. Since q' is a principal over $A\bar{c}_0$, there exists $\theta(\bar{x}, \bar{c}_0)$, an $A\bar{c}_0$-formula , such that $q'(M, \bar{c}_0) = \theta(M, \bar{c}_0)$ and therefore, $q'(M, \bar{c}_0) \subset q(M)$, what contradicts to the condition of almost orthogonality of the types p and q.

Let every extension of the type q over A and any realization of the type p is a non-principal type. We will obtain a contradiction by supposing that $p \not\perp^a q$. From the last it follows that there exist $\bar{c}_0 \vDash p$ and an $A\bar{c}_0$-formula $\varphi(\bar{x}, \bar{c}_0)$ such that

$\varphi(M, \bar{c}_0) \subset q(M)$. Then by the point 2) it follows that there exists an isolating subformula $\varphi_0(M, \bar{c}_0) \subset \varphi(M, \bar{c}_0) \subset q(M)$. Then for some principal type q' we have the following:

$$\varphi_0(M, \bar{c}_0) = q'(M, \bar{c}_0) \subset q(M),$$

what is a contradiction, since any extension of the type q over any realization of p is a non-principal type.

4) If $tp(\bar{c}/\bar{b}) \perp^a q(\bar{x}, \bar{b})$, then every type $q(\bar{x}, \bar{b}) \subset q'(\bar{x}, \bar{b}\bar{c})$ is non-principal. By the condition 3) $th(\bar{d}/\bar{c}\bar{b}) \perp aq'(x/\bar{c}\bar{b})$ for every $q'(x/\bar{c}\bar{b})$ extending $q'(x, \bar{b})$. The last means that every $q''(x/\bar{b}\bar{c}\bar{d})$ extending $q'(x, \bar{b})$ will be non-principal. Then $tp(\bar{d}\bar{c}/\bar{b}) \perp^a q(\bar{x}\bar{b})$. Let us suppose that $tp(\bar{d}\bar{c}/\bar{b}) \not\perp^a q(\bar{x}\bar{b})$. Then there is a formula $\theta(M, \bar{b}\bar{d}, \bar{c}) \subset q(M, \bar{b})$, and by the condition 2), $\theta(\bar{x}, \bar{b}\bar{d}, \bar{c})$ can be considered to be isolating for some principal type $q''(\bar{x}/\bar{b}\bar{d}\bar{c})$. That is, $q''(\bar{x}/\bar{b}\bar{d}\bar{c})$ is a principal type, what contradicts to the obtained condition that all the types q'' are non-principal.

5) In fact, we used the condition of almost orthogonality while formulating the point 5) in the following sense: $tp(\bar{c}/\bar{b}) \perp^a q(x, \bar{b})$ implies $\psi_0(N, \bar{b}, \bar{c}) \cap q(N, \bar{b}) = \emptyset$.

□

Proposition 4.2 *If p and q are two types from $S(T)$ with $p \not\perp^a q$. Then, if some model $\mathfrak{M} \vDash T$ realize the type p, then the type q is also realized in $\mathfrak{M}$.*

Proof of Proposition 4.2 The Proposition states, that the realization of p in some model of T implies the realization of q in the same model. In other terms, p is powerful over q.

If p is realized in some structure $\mathfrak{M} \vDash T$, there exists an element $\bar{\alpha} \in p(M)$. By the definition of an almost orthogonality, there is such a formula $\varphi(\bar{x}, \bar{y})$, that $\emptyset \neq \varphi(M, \bar{\alpha}) \subset q(M)$, what means, that $q(M)$ is not empty, therefore q is realized in $\mathfrak{M}$.

□

The type $r \in S(T)$ is called to be dominated by a type $t \in S(T)$, or in other words r does not exceed the type t, by the **Rudin-Keisler preorder**, denoted as $r \leq_{RK} t$, if $\mathfrak{M}_t \vDash r$, that is, $\mathfrak{M}_r$ is an elementary submodel of $\mathfrak{M}_t$.

If $p \not\perp^a q$, then $q \leq_{RK} p$. In small theories these two notions coincide.

As in the previous section let us denote

$$\Lambda_p := \{D | \text{ there is a model } \mathfrak{M} \in Mod(T), \text{ for which } \mathfrak{M} \vDash p, \mathcal{D}(M) = D\}.$$

Then, if $p \not\perp^a q$, $\Lambda_p \subseteq \Lambda_q$.

Theorem 4.1 [62, P. 53] *Let we have a countable small theory T, and let $\{r_i | i < \omega\}$ be a countable set of all non-isolated types from $S(T)$, then*
1) If for every $r_i \neq r_j$, $r_i \perp^a r_j$, then $I(T, \omega) \geq \omega$.
2) If for every finite subset $\{r_{i_1}, \dots, r_{i_n}\}$ of $\{r_i\}$

27

$$(r_{i_1} \cup r_{i_2} \cup \ldots \cup r_{i_n}) \perp^a r_k, \ k < \omega,$$

then $I(T, \omega) = 2^\omega$.

Proof of Theorem 4.1 1) Take an arbitrary $n \in \mathbb{N}$. Since $r_n \perp^a r_i$, $\forall i \neq n$, $i < \omega$, realization of r_n does not imply realizations of r_i. Therefore, by the Theorem 0.1, there exists a model $\mathfrak{M}_n$ realizing the only type r_n.

Since $\forall n, m \in \mathbb{N}$ $n \neq m$ implies $\mathfrak{M}_n \not\cong \mathfrak{M}_m$, there exists at least countable number of models.

2) Let τ be a countable sequence of 0's and 1's. Divide $\{r_i\}$ into two ordered sets of types, namely, $\{p_i\}$ and $\{q_i\}$, for which $r_i = p_k$ if $\tau(i) = 0$ and $r_i = q_k$ if $\tau(i) = 1$.

For every $n \in \mathbb{N}$ let us take the finite parts of the sets $\{p_i\}$ and $\{q_i\}$: $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$. Take a prime model $\mathfrak{M}_n$ over any extension of $(p_1 \cup \ldots \cup p_n)$. Since $(p_1 \cup \ldots \cup p_n) \perp^a q_i$, $1 \leq i \leq n$, their realization does not imply realization of the types q_i, $1 \leq i \leq n$. Therefore, there the model $\mathfrak{M}_n$ will realize all p_i, and omit all q_i, $i \leq n$. Then, we can construct a model $\mathfrak{M}$, realizing all p_i and omitting all q_i.

By construction, for every τ there there exists a model $\mathfrak{M}_\tau$. And all these models are not isomorphic, since they differ in at least one type. Therefore, $I(T, \omega) = 2^\omega$.

$\square$

Theorem 4.2 [62, P. 54] *If the countable theory T is small, and the countable set $\{r_i \in S(T) \mid i < \omega\}$ of all non-isolated types with $r_i \not\perp^a r_{i+1}$ and $r_{i+k} \perp^a r_i$, then the number of non-isomorphic countable models of T at least will be countable, that is, $I(T, \omega) \geq \omega$.*

Proof of Theorem 4.2 We will construct a model $\mathfrak{M}_n$ for every natural number n. If a model $\mathfrak{M}_i$ realize a type r_i, then by the Proposition 4.2 it realizes all the types r_j, $j > i$.

The model $\mathfrak{M}_1$ realize a type r_1 and, consequently, all the types r_i, $i < \omega$. In the model $\mathfrak{M}_i$, by the Omitting Types Theorem, the types r_j, $j < i$ are omitted, and, since $r_{l+k} \perp^a r_l$, the types r_j, $j \geq i$ are realized.

$\square$

5 LINEAR ORDERS AND EXTREME TRIVIALITY

In this section, like in study of o-minimality, we restrict to theories whose models are linearly ordered. But rather than the global hypothesis that all definable subsets are definable with just the order, we posit conditions on particular types and on the underlying linear order which imply the existence of continuum many countable models.

In the article [53, P. 392] M. Rubin investigated theories of pure linear orders and their expansions using finite and countable sets of unary predicates. He proved that such a theory T has the countable spectrum to be either finite or 2^ω, and in case of finiteness of the language of T is finite, then T is either ω-categorical, or it has the maximal number of countable non-isomorphic model. Thus M. Rubin solved the Vaught Conjecture for linear orders expanded by unary predicates. In our results there will be no restriction on language.

Further in the section as usual we will consider small theories. Given a finite subset $A \subseteq M$ of a model $\mathfrak{M} \vDash T$, we will denote $T(A) := Th(\mathfrak{M}, a)_{a \in A}$. Note that if T is a small theory, then the $T(A)$ is a small theory as well. Also the condition of T being small implies existence of a prime model, $\mathfrak{M}(A)$, of T over the finite set A, and of a countably saturated model of T. If $\bar{a}_1$, $\bar{a}_2$, ..., $\bar{a}_n \in M$, $n \geq 1$, are some tuples of elements of M, then $M(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$ will mean a prime model of T over the set of all elements belonging to those tuples.

5.1 Variants of triviality

Definition 5.1.1 [63] *Let T be a small complete theory, $p(\bar{x})$ be a non-principal type over a finite subset A of some model of T.*

1) The type p is called **extremely trivial**, *if for every natural number $n \geq 1$ and every sequence $\bar{\beta}_1$, $\bar{\beta}_2$, ...$\bar{\beta}_n$ of elements realizing the type p, we have that $p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})) = \{\bar{\beta}_1, \bar{\beta}_2 \ldots \bar{\beta}_n\}$, where $\bar{a}$ is some enumeration of the set A.*

2) The type p is **almost extremely trivial**, *if for every $n \geq 1$ and every sequence $\bar{\beta}_1, \bar{\beta}_2, \ldots \bar{\beta}_n$ realizing the type p, $p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}))$ is finite.*

3) The type p is said to be **eventually extremely trivial**, *if for every natural number $n \geq 1$ there is $m \geq n$ and realizations $\bar{\beta}_1, \bar{\beta}_2, \ldots \bar{\beta}_m$ of the type p for which $p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m, \bar{a})) = \{\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\}$.*

It is easy to see that every extremely trivial type is almost extremely trivial, and every almost extremely trivial type is eventually extremely trivial.

Example 5.1.1 [63, P. 720]   Let $\mathcal{L} = \{=, P_i\}_{i < \omega}$, where the P_i are unary, and let T be an $\mathcal{L}$-theory and that the P_i are a decreasing sequence of sets with each $P_i - P_{i+1}$ infinte. It can be axiomtized as follows.
   1) $\forall x \ (P_{i+1}(x) \to P_i(x))$ for all i < ω; and
   2) $\exists^{\geq n} x \ (P_i(x) \land \neg P_{i+1}(x))$ for every natural number n < ω, and $i < \omega$.

Then the type $p(x) := \{P_i(x) \mid i < \omega\}$ is extremely trivial, and the theory T has $\aleph_0$ countable models.

Example 5.1.2 [63, P. 720]   Let $\mathcal{L} = \{=, P_i, R\}_{i<\omega}$ with the P_i unary and R binary, $k \geq 2$ be an integer, and T_k be an $\mathcal{L}$-theory that asserts the P_i's are a descending sequence of definable sets and R is a relation of equivalence with infinitely many classes, all of cardinality k and such no equivalence class can be split by a P_i. Axioms:

1) $\forall x \quad (P_{i+1}(x) \rightarrow P_i(x))$ for every $i < \omega$;
2) $\exists^{\geq n} x \quad (P_i(x) \wedge \neg P_{i+1}(x))$ for every natural $n < \omega$, $i < \omega$;
3) $\forall x \quad R(x, x)$;
4) $\forall x \forall y \quad (R(x, y) \rightarrow R(y, x))$;
5) $\forall x \forall y \forall z \quad ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$;
6) $\forall x \exists^{=k} y \quad R(x, y)$; and
7) $\forall x \forall y \quad ((R(x, y) \wedge P_i(x)) \rightarrow P_i(y))$ for every $i < \omega$.

Let $p(x) := \{P_i(x) \mid i < \omega\}$. The type $p(x)$ is almost extremely trivial, but is not extremely trivial. This theory has $\aleph_0$ countable models: for every natural number n, a model with exactly kn realizations of p.

Example 5.1.3 [63, P. 721]   Let $\mathcal{L} = \{=; <; P_i\}_{i<\omega}$, with the P_i unary and T be an $\mathcal{L}$-theory axiomatized by the following:

1) $<$ is a dense linear order without endpoints;
2) P_i's are dense codense disjoint predicates.

The type $p(x) := \{\neg P_i(x) \mid i < \omega\}$ is extremely trivial. This theory has $2^{\aleph_0}$ countable non-isomorphic models.

The following example including a unary function shows that our results extend those of M. Rubin [53, P. 392].

Example 5.1.4 [63, P. 721] Modify Example 5.1.3 by adding a constant symbol 0 and a unary function f satisfying $f^2(x) = x$, $f(0) = 0$ and $x > y > 0$ implies $f(x) < f(y) < 0$.

The type $p(x) := \{\neg P_i(x) \quad \mid \quad i < \omega\}$ is extremely trivial. By Theorem 5.2.3 this theory has $2^{\aleph_0}$ countable non-isomorphic models.

Definition 5.1.2 [63, P. 721] *1) An A-definable formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a})$ with $\bar{a}$ and element of A, is called a **p-n-preserving** formula, if for every sequence $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n$ realizing the type p, $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}) \vdash p(\bar{x})$.*
*2) If $q(\bar{y}_1, \ldots, \bar{y}_n)$ $(n < \omega)$ is an A-type with $\bigcup_{1 \leq i \leq n} p(\bar{y}_i) \cup \{\bigwedge_{1 \leq i \neq j \leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$. An A-definable formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a})$, $\bar{a} \in A$, is called **p-q-preserving**, if*

for every sequence $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$ realizing the type p, we have: $tp(\bar{\beta}_1, \ldots, \bar{\beta}_n) = q$ implies $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}) \vdash p(\bar{x})$.

*3) A p-n-preserving (p-q-preserving) formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a})$ is **non-trivial**, if for every model $\mathfrak{M} \vDash T$ and every realizations $\bar{\beta}_i$, $1 < i < n$, of the type p in $\mathfrak{M}$ (with $tp(\bar{\beta}_1, \ldots, \bar{\beta}_n/A) = q$) the set $\varphi(M, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})$ contains at least one element other than $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n$.*

Proposition 5.1.1 [63, P. 721] *Let the theory T be countable and complete, $p(\bar{x}) \in S(A)$ be a non-principal type over a finite subset A of some model of T. Then the type p is extremely trivial if and only if for every $n \geq 1$ every p-n-preserving A-definable formula is trivial.*

Proof of Proposition 5.1.1 Further by $\bar{a}$ we will denote a tuple enumerating the set A.

($\Rightarrow$) Let p be extremely trivial, $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$ ($n \geq 1$) be realizations of p, and $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a})$ be a p-n-preserving A-definable formula. Directly from the definitions it follows that $\varphi(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}), \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}) \subseteq p((M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})) = \{\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n\}$.

Therefore, the formula φ is trivial.

($\Leftarrow$) Now suppose that for every $n \geq 1$ every p-n-preserving A-definable formula is trivial. Take a finite number of arbitrary realizations of p, namely, $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$. Towards a contradiction let us suppose that there exists a realization $\bar{\beta} \in p((M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}))$ other than $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$. Let $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})$ be the formula isolating the principal type $p'(\bar{x}) := tp(\bar{\beta}/\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})$. Since $p(x) \subseteq p'(x)$, φ is p-n-preserving. And since $(\bigwedge_{1 \leq i \leq n} \bar{x} \neq \bar{\beta}_i) \in p'(x)$, φ is non-trivial. This is a contradiction.

$\square$

Proposition 5.1.2 [63, P. 722] *Let we are given a countable complete theory T, let $p(\bar{x}) \in S(A)$ be a non-principal type over a finite subset A of a model of T. Then the following statements are equivalent:*

1) The type p is almost extremely trivial;

2) For every $n \geq 1$, and for every A-type $q(\bar{y}_1, \ldots, \bar{y}_n)$ with $\bigcup_{1 \leq i \leq n} p(\bar{y}_i) \cup \{\bigwedge_{1 \leq i \neq j \leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, there exists no more than finite number of non-equivalent non-trivial p-q-preserving A-formulas, and for every realizations $\bar{\beta}_1, \ldots, \bar{\beta}_n$ with $tp(\bar{\beta}_1, \ldots, \bar{\beta}_n/A) = q$, and every p-q-preserving A-formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a})$, the formula $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})$ is algebraic;

3) For every $n \geq 1$, and every A-type $q(\bar{y}_1, \ldots, \bar{y}_n)$ with $\bigcup_{1 \leq i \leq n} p(\bar{y}_i) \cup \{\bigwedge_{1 \leq i \neq j \leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, there exist $m \geq n$ and a type $q'(\bar{y}_1, \ldots, \bar{y}_m) \supseteq q$ such that for each $\bar{\beta}_1, \ldots, \bar{\beta}_m \vDash q'$, $p(M(\bar{\beta}_1, \ldots, \bar{\beta}_m, \bar{a})) = \{\bar{\beta}_1, \ldots, \bar{\beta}_m\}$.

Proof of Proposition 5.1.2 Further by $\bar{a}$ we denote some tuple enumerating the set A.

1) $\Rightarrow$ 2) Let p be almost extremely trivial. Let $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a})$ be a non-trivial p-q-n-preserving A-definable formula ($n \geq 1$), where $q(\bar{y}_1, \ldots, \bar{y}_n)$ is some A-type with $\bigcup_{1 \leq i \leq n} p(\bar{y}_i) \cup \{ \bigwedge_{1 \leq i \neq j \leq n} \bar{y}_i \not\approx \bar{y}_j \} \subseteq q$, and $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$ be some realizations of p. Since $\varphi(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}), \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}) \subseteq p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}))$, and p is almost extremely trivial, this set is finite, and $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})$ is an algebraic formula.

Now towards a contradiction suppose that there exist $n \geq 1$, an A-type $q(\bar{y}_1, \ldots, \bar{y}_n)$ with $\bigcup_{1 \leq i \leq n} p(\bar{y}_i) \cup \{ \bigwedge_{1 \leq i \neq j \leq n} \bar{y}_i \not\approx \bar{y}_j \} \subseteq q$, and an infinite family Φ of pairwise non-equivalent non-trivial p-q-preserving A-definable formulas. Let us take any n realizations, $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$, of q. For every $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a}) \in \Phi$ the we have $\varphi(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}), \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}) \subseteq p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}))$.

Since the set Φ is infinite, and all the formulas from Φ are pairwise non-equivalent, $p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}))$ should be infinite, what is impossible because of almost extreme triviality of p.

2) $\Rightarrow$ 3) Let n and q be as in 3), and $\bar{\beta}_1$, ..., $\bar{\beta}_n$ be realizations of q. If every p-q-preserving formula is trivial, then the desired type q' is q itself, and the proof is done. If not, then let us take an arbitrary element $\bar{\gamma} \in p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})) \setminus \{\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n\}$. Denote by $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})$ an isolating formula of the principal type $tp(\bar{\gamma}/\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})$. Since $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}) \vdash p$, for every formula $\psi(\bar{x}, \bar{a}) \in p$ we have $\vDash \forall \bar{x}(\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}) \to \psi(\bar{x}, \bar{a}))$. And therefore, the formula $\forall \bar{x}(\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a}) \to \psi(\bar{x}, \bar{a}))$ belongs to the type $tp(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})$. Since the last holds for every formula $\psi(\bar{x}, \bar{a})$ from the type p, we have that the formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{a})$ is non-trivial p-q-preserving. By 2) this formula is algebraic, and then the set $\varphi(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}), \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}) \subseteq p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}))$ is finite. This holds for every element $\bar{\gamma} \in p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a})) \setminus \{\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n\}$, and since by 2) there exists only finite number of non-equivalent non-trivial p-q-preserving formulas, the set $p(M(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{a}))$ is finite, and is equal to $\{\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\}$, where $m > n$, and $\bar{\beta}_i \vDash p$ for all i, $n < i \leq m$. Denote by q' the type $tp(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m/\bar{a})$, it is easy to see that q' is the desired type.

3) $\Rightarrow$ 1) Now let we have an arbitrary $n \geq 1$ and realizations $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n$ of p. Let us denote by q the type $tp(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n/\bar{a})$. By 3) there are $m \geq n$ and a type $q'(\bar{y}_1, \ldots, \bar{y}_m)$ containing the type q, such that $p(M(\bar{\beta}'_1, \ldots, \bar{\beta}'_m, \bar{a})) = \{\bar{\beta}'_1, \ldots, \bar{\beta}'_m\}$ for every realizations $\bar{\beta}'_1$, ..., $\bar{\beta}'_m \vDash q'$ If we have $m = n$, then the proof for this n is finished. Now take arbitrary $\bar{\beta}_{n+1}$, $\bar{\beta}_{n+2}$, ..., and $\bar{\beta}_m$, realizations of the type p for which $\bar{\beta}_i \neq \bar{\beta}_j$ for all $1 \leq i \leq n$ and $n + 1 \leq j \leq m$. Then the following holds: $p(M(\bar{\beta}_1, \ldots, \bar{\beta}_n, \bar{a})) \subseteq p(M(\bar{\beta}_1, \ldots, \bar{\beta}_m, \bar{a})) = \{\bar{\beta}_1, \ldots, \bar{\beta}_m\}$, Therefore $p(M(\bar{\beta}_1, \ldots, \bar{\beta}_n, \bar{a}))$ is finite, and, since the proof is done for arbitrary n, p is an almost extremely trivial type.

□

The following can be obtained as an easy corollary of the proof of Proposition 5.1.2.

Proposition 5.1.3 [63, P. 723] *Let we are given a countable complete theory T, and a non-principal type $p(\bar{x}) \in S(A)$ over a finite subset A of some model of T. Then the next statements are equivalent:*

1) The type p is eventually extremely trivial;

2) For every $n \geq 1$, there exist m ($n \leq m$), and an A-type $q(\bar{y}_1, \ldots, \bar{y}_m)$ such that $\bigcup\limits_{1 \leq i \leq n} p(\bar{y}_i) \cup \{\bigwedge\limits_{1 \leq i \neq j \leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, there exists no more than finite number of non-equivalent non-trivial p-q-preserving A-formulas, and for every $\bar{\beta}_1, \ldots, \bar{\beta}_m$ with $tp(\bar{\beta}_1, \ldots, \bar{\beta}_m / A) = q$, for every p-q-preserving A-formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m, \bar{a})$ the formula $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m, \bar{a})$ is algebraic;

3) For every $n \geq 1$, there is such an A-type $q(\bar{y}_1, \ldots, \bar{y}_n)$ for which $\bigcup\limits_{1 \leq i \leq n} p(\bar{y}_i) \cup \{\bigwedge\limits_{1 \leq i \neq j \leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, there exist $m \geq n$ and a type $q'(\bar{y}_1, \ldots, \bar{y}_m) \supseteq q$ such that for every $\bar{\beta}_1, \ldots, \bar{\beta}_m \vDash q'$, $p(M(\bar{\beta}_1, \ldots, \bar{\beta}_m, \bar{a}))$ coincides with $\{\bar{\beta}_1, \ldots, \bar{\beta}_m\}$.

5.2 Number of countable models

Theorem 5.2.1 [63, P. 723] *Let we are given a small complete theory T. If there exists a finite subset A of some model of T and an eventually extremely trivial non-isolated type $p(\bar{x}) \in S(A)$, then $I(T \cup tp(\bar{a}/\emptyset), \omega) \geq \omega$, where $\bar{a}$ is a tuple enumerating the set A.*

Proof of Theorem 5.2.1 Since the type p is eventually extremely trivial, there are $m_1 \geq 1$ and m_1 realizations $\bar{\beta}_1, \bar{\beta}_1, \ldots \bar{\beta}_{m_1}$ of p which are the only realizations of p in $\mathfrak{M}_1 := \mathfrak{M}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_{m_1}, \bar{a})$, the prime model over realizations. Then we have $|p(M_1)| = m_1$. Analogically, for every $i \geq 1$ there exists a model $\mathfrak{M}_i$ (prime over a finite set) with $|p(M_i)| = m_i \geq m_{i-1} + 1$. Since all those models are not isomorphic, there exists at least countable number of models of $I(T \cup tp(\bar{a}/\emptyset))$.

□

Definition 5.2.1 [64] *Let $\mathfrak{M}$ be a linearly ordered structure, $A \subseteq M$, M be $|A|^+$-saturated, and $p \in S_1(A)$ be a non-algebraic type.*

*1) An A-definable formula $\varphi(x, y)$ is **p–stable** if there exist α, γ_1, $\gamma_2 \in p(M)$ for which $\gamma_1 < \varphi(\alpha, M) < \gamma_2$ and $p(M) \cap [\varphi(\alpha, M) \backslash \{\alpha\}] \neq \emptyset$.*

*2) A p-stable formula $\varphi(x, y)$ is called **convex to the right (left)** if there exists such a realization $\alpha \vDash p$ that α is the left (or right) endpoint of the set $\varphi(\alpha, M)$, $\alpha \in \varphi(\alpha, M)$ and $p(M) \cap \varphi(\alpha, M)$ is a convex set.*

*3) A p-stable convex to the right (to the left left) formula $\varphi(x, y)$ is said to be a **quasi-successor** on the type p if for every realization $\alpha \in p(M)$ there is $\beta \in \varphi(\alpha, M) \cap p(M)$ with $p(M) \cap [\varphi(\beta, M) \backslash \varphi(\alpha, M)] \neq \emptyset$.*

In the section 7 we will return to the notion of a quasi-successor formula, and we will prove the following theorem:

Theorem 5.2.2 [65] *Let we are given a theory T of (an expansion of) linear order, let A be a finite subset of some model of T, and $p(x)$ be a 1-type over A. Then if there exists an A-definable formula quasi-successor on the type p, then the theory T has $2^{\aleph_0}$ nonisomorphic countable models.*

Lemma 5.2.1 *Suppose that we are given small complete theory T of (an expansion of) a linear order, which has less than the maximal number of countable non-isomorphic models. Also let A be a finite subset of some model of the theory T, and $p(x) \in S_1(A)$ be a non-principal 1-type over the set A. Then for any two elements α, β realizing the type p, $\{\alpha < x < \beta\} \cup p(x)$ is a consistent set of formulas.*

Proof of Lemma 5.2.1 Towards a contradiction let us assume the contrary. Then there exists a finite subset $\Phi \subset p(x)$ which is inconsistent with the formula $\{\alpha < x < \beta\}$ in T. Denote $\theta(x, \bar{a}) := \bigwedge_{\varphi \in \Phi} \varphi(x)$.

Take a countable saturated model $\mathfrak{M} \vDash T$ with $\alpha, \beta \in M$, and $A \subset M$. By our assumption we have $\mathfrak{M} \vDash \neg \exists x \quad (\alpha < x < \beta \wedge \theta(x, \bar{a}))$.

Now let us take an elementary monomorphism which maps α to β. This monomorphism can be extended to an automorphism $f \in Aut_A(\mathfrak{M})$. Since $\alpha < \beta$, we have $\beta = f(\alpha) < f(\beta)$, and analogically: $f^n(\beta) < f^{n+1}(\beta)$, for every $n \in \mathbb{Z}$. Therefore the set $\theta(M, \bar{a})$ contains an infinite discretely ordered chain

On the set $\theta(M, \bar{a})$ we introduce a binary relation $<^*$, defined by the following formula: $x <^* y := x < y \wedge \theta(x, \bar{a}) \wedge \theta(y, \bar{a}) \wedge \neg \exists z(\theta(z, \bar{a}) \wedge x < z < y)$.

Consider the following set of formulas:

$$p(x) \cup p(y) \cup \{x < y \wedge \forall z((x < z < y \wedge \theta(z, \bar{a})) \to \exists u_1 \exists u_2 (\theta(u_1, \bar{a}) \wedge$$
$$\theta(u_2, \bar{a}) \wedge x < u_1 <^* z <^* u_2 < y))\} \cup \{\exists u_1 \exists u_2 \dots \exists u_n (\bigwedge_{1 \leq i \leq n} \theta(u_i, \bar{a}) \wedge x <$$
$$u_1 <^* u_2 <^* \dots <^* u_n < y)\}.$$

This set is consistent, therefore, it can be completed to a 2-type over A. Fix some realization, γ_1, γ_2, of the obtained type in the model $\mathfrak{M}$.

Let $r(x)$ be a completion of the formula $\gamma_1 < x < \gamma_2$ to a type over $A \cup \{\gamma_1, \gamma_2\}$.

Then the formula $\varphi(x, y, \bar{a}) := x = y \vee x <^* y$ is a quasi-successor on r.

Therefore by Theorem 5.2.2 $T \cup tp(\alpha, \beta, \gamma_1, \gamma_2, \bar{a})$ has the maximal number of countable models up to an isomorphism. Since every model of the given theory T has no more than ω countable models of $T \cup tp(\alpha, \beta, \gamma_1, \gamma_2, \bar{a})$, then $I(T, \omega) = 2^{\aleph_0}$, which is a contradiction with the statement of the theorem.

$\square$

Lemma 5.2.2 [63, P. 724] *Let $\mathfrak{M}$ be a structure of a countable small complete*

theory T, *where* A *and* D *be finite subsets of* M, *and* B *is a countable subset of* M. *For each* $(A \cup B \cup D)$-*formula*, $\varphi(x, \bar{a}, \bar{b}, \bar{d})$, *where* $\bar{a}$ *enumerates the set* A, $\bar{b} \in B$, *and* $\bar{d} \in D$, *there exists a type* $q_\varphi = q \in S_1(A \cup B \cup D)$ *such that*

1) $\varphi(x, \bar{a}, \bar{b}, \bar{d}) \in q$;

2) The set B *can be written as union of finite subsets* B_n *such that for every* n, q/B_n *is principal.*

Proof of Lemma 5.2.2 Enumerate B as $\{b_1, b_2, \ldots, b_i, \ldots\}$. For $i < \omega$ denote $\bar{b}_i := \langle b_1, b_2, \ldots b_i \rangle$, and let $\bar{d}'$ be a tuple enumerating the set D. Because the theory T is small, there exists a formula $\varphi_0(x, \bar{a}, \bar{b}_n, \bar{d}')$ that implies $\varphi(x, \bar{a}, \bar{b}_n, \bar{d})$ and generates a principal type over $(A \cup \{\bar{b}_n\} \cup D)$. In turn there is a principal subformula over $(A \cup \{\bar{b}_{n+1}\} \cup D)$ that implies $\varphi_0(x, \bar{a}, \bar{b}_n, \bar{d}')$. Repeating this construction, we will get a consistent infinite chain of decreasing principal over parameters formulas $\varphi_i(x, \bar{a}, \bar{b}_{n+i}, \bar{d}')$: $\ldots$ $\subseteq \varphi_{i+1}(N, \bar{a}, \bar{b}_{n+i+1}, \bar{d}') \subseteq$ $\varphi_i(N, \bar{a}, \bar{b}_{n+i}, \bar{d}') \subseteq$ $\ldots$ $\subseteq$ $\varphi_0(N, \bar{a}, \bar{b}_n, \bar{d}') \subseteq \varphi(N, \bar{a}, \bar{b}_n, \bar{d})$, where $\mathfrak{N}$ is an arbitrary model of T with $(A \cup B \cup D) \subseteq N$. Let $\bar{b}_n$ enumerate B_n, we have defined the desired complete type over $(A \cup B \cup D)$.

$\square$

Theorem 5.2.3 [63, P. 724] *Let* T *be a countable complete theory of (an expansion of) linear order. If there exists a finite subset* A *of a model* $\mathfrak{M} \vDash T$ *and exists a type* $p(x) \in S_1(A)$ *which is non-principal and extremely trivial, then the theory* T *has* $2^{\aleph_0}$ *countable models up to an isomorphism.*

Proof of Theorem 5.2.3 Since every theory which is not small has $2^{\aleph_0}$ countable non-isomorphic models, it remains to prove the case, when the theory T is small.

Denote by $\mathfrak{N}$ an $\aleph_1$-saturated elementary extension of $\mathfrak{M}$.

During the proof, we will construct a countable model $\mathfrak{M}_\tau \prec \mathfrak{N}$ for every infinite sequence of zeros and ones, $\tau := \langle \tau(1), \tau(2), \ldots, \tau(i), \ldots \rangle_{i<\omega}$, $\tau(i) \in \{0,1\}$, such that for every $\tau_1 \neq \tau_2$, $\mathfrak{M}_{\tau_1} \not\cong \mathfrak{M}_{\tau_2}$.

Let us fix such a sequence of zeros and ones, τ.

Denote by $\mathbb{Q}_\tau$ the following subset of rational numbers:

$$\mathbb{Q}_\tau := \bigcup_{n \geq 0}(2n, 2n+1) \cup \bigcup_{\substack{n \geq 1, \\ \tau(n)=0}} \{2n - \tfrac{1}{3}, 2n - \tfrac{2}{3}\} \cup \bigcup_{\substack{n \geq 1, \\ \tau(n)=1}} \{2n - \tfrac{1}{5}, 2n - \tfrac{2}{5}, 2n - \tfrac{3}{5}\}.$$

Now, pick from the set $p(N)$ a subset, ordered by the type of $\mathbb{Q}_\tau$. If such a subset does not exist, then by Lemma 5.2.1 T has $2^{\aleph_0}$ countable models, and the theorem is proved. Denote this subset by $B := \{b_1, b_2, \ldots, b_i, \ldots\}_{i<\omega}$. Also, for each $n < \omega$ let $\bar{b}_n$ be the tuple $\langle b_1, b_2, \ldots, b_n \rangle$. For the model $\mathfrak{M}_\tau$ we will have $p(\mathfrak{M}_\tau) = B$.

Using the Tarski-Vaught test we will show that the set M_τ is a universe of some elementary substructure of $\mathfrak{N}$. On each step of the construction we will be fixing a set of parameters and promising to realize each satisfiable 1-formula over it. We must keep

coming back to the same set of parameters and deal with another formula. So the different sets of parameters are being attacked in parallel. We will choose the realizations in a certain way, which, together with extreme triviality of the type p, will imply that the only realizations of this type will be the elements of the set B.

Step 1. Let us denote by Φ_1 the set of all A-definable 1-formulas, $\Phi_1 := \{\varphi_i^1(x, \bar{a}) | i < \omega\}$, where $\bar{a}$ is a tuple enumerating the set A. Choose the least i such that $\mathfrak{N} \models \exists x \varphi_i^1(x, \bar{a})$. To satisfy the Tarski-Vaught property, we must find a witness for $\varphi_i^1(x, \bar{a})$. Since the sets A, B and the formula φ_i^1 are as in Lemma 5.2.2 (consider the set D to be empty), there exists an $A \cup B$-type $q_{\varphi_i^1}$ satisfying conditions 1) and 2) from the lemma. And since the model $\mathfrak{N}$ is $\aleph_1$-saturated, this type is realized in $\mathfrak{N}$ by some element, denote it by d_1. Therefore the element d_1 is principal over the set A.

Step 2. Let us take smallest index j for which $\varphi_j^1(x, \bar{a}) \in \Phi_1$ was not taken before and the following holds: $\mathfrak{N} \models \exists x \varphi_j^1(x, \bar{a})$. We find a special witness for $\varphi_j^1(x, \bar{a})$, which will satisfy the Tarski-Vaught condition but not realize p. By applying the Lemma 5.2.2 to the sets A, B and $\{d_1\}$, and $\varphi_j^1(x, \bar{a})$, we can choose d_2, a realization of $q_{\varphi_j^1}$. The element d_2 can be chosen to be principal over the set Ab_1d_1.

Now let us take the element b_1 and construct the set of $(A \cup \{b_1\} \cup \{d_1\})$-definable unary formulas, which we denote by $\Phi_2 := \{\varphi_i^2(x, \bar{a}, b_1, d_1) | i < \omega\}$. Now choose the least index i for which $\varphi_i^2(x, \bar{a}, b_1, d_1)$ from the family Φ_2 was not chosen before, and $\mathfrak{N} \models \exists x \varphi_i^2(x, \bar{a}, b_1, d_1)$, and find a realization d_3 existing by the Lemma 5.2.2 applied to the sets A, B, $\{d_1, d_2\}$, and the formula φ_i^2.

At the stage k the next sets would be chosen:

– Nested sets $D_1 = \{d_1\}$, $D_2 = \{d_1, d_2, d_3\}$, $D_3 = \{d_1, d_2, \ldots, d_6\}$, ..., $D_k = \{d_1, d_2, \ldots, d_{\frac{(k+1)k}{2}}\}$, where D_i was constructed on step i through adding to D_{i-1} of i new realizations. For some i and j we might have $d_i = d_j$, where $1 \leq i < j \leq \frac{(k+1)k}{2}$.

– The family of all A-definable 1-formulas Φ_1, and for every m, $2 \leq m \leq k$, a family of $(A \cup \{\bar{b}_{m-1}\} \cup D_{m-1})$-definable 1-formulas, Φ_m.

Further we will use the usual notation $\bar{d}_i = \langle d_1, d_2, \ldots, d_i \rangle$, $i < \omega$.

Step $k + 1$. Firstly, we realize one formula from each of the families we defined earlier. To do this, for each m, $1 \leq m \leq k$, find smallest number i_m for which formula $\varphi_{i_m}^m \in \Phi_m$ were not taken previously, and the definable set in $\mathfrak{N}$ of which is not empty. Apply Lemma 5.2.2 to the sets A, B and $\{\bar{d}_{\frac{(k+1)k}{2}+m-1}\}$, and the formula $\varphi_{i_m}^m$, to find realization $d_{\frac{(k+1)k}{2}+m}$ of the type $q_{\varphi_{i_m}^m}$.

Now let Φ_{k+1} be the set of all $(A \cup \{\bar{b}_k\} \cup D_k)$-definable 1-formulas, find the smallest index i such that $\mathfrak{N} \models \exists x \varphi_i^{k+1}(x, \bar{a}, \bar{b}_k, \bar{d}_{\frac{(k+2)(k+1)}{2}})$. And choose $d_{\frac{(k+1)k}{2}+k+1}$ as before, as a realization of a type $q_{\varphi_i^{k+1}}$, which exists by Lemma 5.2.2 applied to the sets A, B, $\{\bar{d}_{\frac{(k+1)k}{2}+k}\}$, and formula φ_i^{k+1}. Let D_{k+1} be the set

36

$\{d_1, d_2, \ldots, d_{\frac{(k+1)k}{2}+k+1}\}$. We can arrange that each new d_i is principal over $A\bar{b}_n$ and the d_j's for $j < i$.

Denote $M_\tau := A \cup B \cup \bigcup_{i<\omega} D_i$.

Suppose that there exists a realization $\delta \in p(N)\backslash B$. Since the type p is not principal, $\delta \notin A$, then for some $k < \omega$, $\delta = d_k$. For every $n < \omega$ the type $tp(d_k/\bar{a}\bar{b}_n)$ is non-principal. Otherwise, it should be realized in $\mathfrak{M}(\bar{a}, \bar{b}_n)$ by some element not from $\bar{b}_n$, which is impossible since the type p is extremely trivial. Also, since for every $i < \omega$, we choose d_i to satisfy the conditions of Lemma 5.2.2, we have that the type $tp(d_i/\bar{a}, \bar{b}_n, \bar{d}_{i-1})$ is principal. From the last statement it easily follows by induction that the type $tp(\bar{d}_k/\bar{a}\bar{b}_n)$ is principal, and therefore $tp(d_k/\bar{a}\bar{b}_n)$ is a principal type as well. This is a contradiction, and we have $p(\mathfrak{M}_\tau) = B$.

The obtained structure $\mathfrak{M}\tau$ is an elementary substructure of the structure $\mathfrak{N}$. This is true because of the Tarski-Vaught test. And have that $I(T \cup tp(\bar{a}), \omega) = 2^{\aleph_0}$ since the number of different sequences τ is equal to the cardinality of the continuum. The theory T, being small, has at most countably many distinct complete extensions by realizing an n-type, $T \cup tp(\bar{a})$; consequently, $I(T, \omega) = 2^{\aleph_0}$.

□

By this we have proved the main theorem of the section, which guarantees that a countable theory which has an extremely trivial 1-type over a finite subset, has the maximal number of countable models up to an isomorphism.

6 MAXIMALITY OF NUMBER OF COUNTABLE MODELS FOR PARTIALLY ORDERED THEORIES

Consider a theory T be countable and complete, and let $\bar{c}$ to be some element of some structure of T.

We generalize the usual concept of a partial order onto a definable order on tuples of elements.

Definition 6.1 [69] *We call a formula $\psi(\bar{x}, \bar{y}, \bar{c})$ with $ln(\bar{x}) = ln(\bar{y})$ to be defining a **partial order** in a theory T, if for any structure $\mathfrak{M} \vDash T$ such that $\bar{c} \in M$ the following holds:*

$\mathfrak{M} \vDash \forall \bar{x} \forall \bar{y} (\psi(\bar{x}, \bar{y}, \bar{c}) \rightarrow \bar{x} \neq \bar{y})$;

$\mathfrak{M} \vDash \forall \bar{x} \forall \bar{y} \neg (\psi(\bar{x}, \bar{y}, \bar{c}) \wedge \psi(\bar{y}, \bar{x}, \bar{c}))$;

$\mathfrak{M} \vDash \forall \bar{x} \forall \bar{y} \forall \bar{z} ((\psi(\bar{x}, \bar{y}, \bar{c}) \wedge \psi(\bar{y}, \bar{z}, \bar{c})) \rightarrow \psi(\bar{x}, \bar{z}, \bar{c}))$.

Definition 6.2 [69, P. 6] *Consider $\psi(\bar{x}, \bar{y}, \bar{c})$ to be a formula which determines a partial order on T, by a **ψ-chain** on the theory T we mean a subset of some structure $\mathfrak{M} \vDash T$ with $\mathfrak{M} \vDash \exists \bar{x} \exists \bar{y} \quad \psi(\bar{x}, \bar{y}, \bar{c})$, which is a linearly ordered by ψ. and is convex (by ψ) in the model $\mathfrak{M}$.*

Let we have a formula $\varphi(\bar{x})$, it may have parameters, by a **convex-$\psi(\bar{x}, \bar{y}, \bar{c})$-closure** of a formula φ we mean the next formula $\varphi^c_{\psi(\bar{x}, \bar{y}, \bar{c})}(\bar{x}) := \exists \bar{y}_1, \exists \bar{y}_2 (\varphi(\bar{y}_1) \wedge \varphi(\bar{y}_2) \wedge ((\psi(\bar{y}_1, \bar{x}, \bar{c}) \vee \bar{x} = \bar{y}_1) \wedge (\psi(\bar{x}, \bar{y}_1, \bar{c}) \vee \bar{x} = \bar{y}_2)))$.

By **convex- $\psi(\bar{x}, \bar{y}, \bar{c})$ -closure** of a type $p(\bar{x})$, we understand the type $p^c_{\psi(\bar{x}, \bar{y}, \bar{c})}(\bar{x}) := \{\varphi^c_{\psi(\bar{x}, \bar{y}, \bar{c})}(\bar{x}) \mid \varphi(\bar{x}) \in p\}$.

Theorem 6.1 [69, P. 6] *Let we are given be a complete countable theory T, and let $\mathfrak{M}$ be a countable structure of the theory T. If there exists a tuple $\bar{c} \in M$, and a sentence $\psi(\bar{x}, \bar{y}, \bar{c})$, $ln(\bar{x}) = ln(\bar{y}) = l$, which defines partial order on the theory T and such that for every natural $n \in \mathbb{N}$ there is a finite discrete ψ-chain of length equal to or more than n, then $I(T, \omega) = 2^\omega$.*

Proof of Theorem 6.1 If the given theory T is not a small theory, then by the theorem 2.3.2 it has the maximal number of countable structures up to an isomorphism, and the theorem is proved. Consequently, later we can consider T to be a small theory.

Firstly note that by the theorem of compactness there exists a discrete ψ-chain of an infinite length.

For simplicity let us denote the following:

$\bar{x} <^* \bar{y} := \psi(\bar{x}, \bar{y}, \bar{c})$;

$\bar{x} \leq^* \bar{y} := \bar{x} <^* \bar{y} \vee \bar{x} = \bar{y}$;

$s(\bar{x}, \bar{y}, \bar{c}) := \bar{x} <^* \bar{y} \wedge \neg \exists \bar{z} (\bar{x} <^* \bar{z} \wedge \bar{z} <^* \bar{y})$;

$s^{(0)}(\bar{x}, \bar{y}) := \bar{x} = \bar{y}$;

$$s^{(n)}(\bar{x}, \bar{y}) := \exists \bar{z}_1 \ldots \exists \bar{z}_n (\bar{z}_1 = \bar{x} \wedge \bar{z}_n = \bar{y} \bigwedge_{i=1}^{n-1} \bar{s}(z_i, \bar{z}_{i+1}));$$

$$s^{(-n)}(\bar{x}, \bar{y}) := \exists \bar{z}_1 \ldots \exists \bar{z}_n (\bar{z}_1 = \bar{x} \wedge \bar{z}_n = \bar{y} \bigwedge_{i=1}^{n-1} \bar{s}(z_{i+1}, \bar{z}_i)), \text{ where } n \in \mathbb{N} \backslash \{0\};$$

$$\varphi(\bar{x})^+ := \exists \bar{y}(\psi(\bar{y}) \wedge \bar{y} <^* \bar{x});$$

$$\varphi(\bar{x})^- := \exists \bar{y}(\psi(\bar{y}) \wedge \bar{x} <^* \bar{y}).$$

For a given natural number we can construct a formula which will determine a discrete chain of length which is more or equal to that number, that is: $\varphi_n(\bar{x}, \bar{y}, \bar{c}) :=$

$$\exists \bar{z}_1 \ldots \exists \bar{z}_n (\bar{z}_1 = \bar{x} \wedge \bar{z}_n = \bar{y} \bigwedge_{i=1}^{n-1} \bar{z}_i <^* \bar{z}_{i+1} \wedge \forall \bar{z}(\bar{x} \leq^* \bar{z} \wedge \bar{z} \leq^* \bar{y} \rightarrow \exists \bar{t}_1 \exists \bar{t}_2 (s(\bar{t}_1, \bar{z}) \wedge s(\bar{z}, \bar{t}_2)))).$$

Denote the type $p(\bar{x}, \bar{y}, \bar{c}) := \{\varphi_n(\bar{x}, \bar{y}, \bar{c}) \mid n < \omega\}$. Let us take the tuple $(\bar{a}, \bar{b}) \vDash p$ and consider the next formula: $\gamma_n(\bar{x}, \bar{a}, \bar{b}, \bar{c}) :=$

$$\exists \bar{x}_1 \ldots \exists \bar{x}_n \exists \bar{y}_1 \ldots \exists \bar{y}_n (\bar{x}_1 = \bar{a} \wedge \bar{y}_1 = \bar{b} \bigwedge_{i=1}^{n-1} (s(\bar{x}_i, \bar{x}_{i+1}) \wedge s(\bar{y}_{i+1}, \bar{y}_1)) \wedge \bar{x}_n <^* \bar{x} \wedge$$

$\bar{x} <^* \bar{y}_1)$ which is a formula meaning that $\bar{x}$ is located between $\bar{a}$ and $\bar{b}$, but it is not an i-th $<^*$-successor (or predecessor) of the tuple $\bar{a}$ $(\bar{b})$ for every $i \leq n$.

Denote $q(\bar{x}, \bar{a}, \bar{b}, \bar{c}) := \{\gamma_n(\bar{x}, \bar{a}, \bar{b}, \bar{c}) \mid n < \omega\}$, this is not necessary a complete type, but a finitely consistent type over $\{\bar{a}, \bar{b}, \bar{c}\}$. Take a countable saturated extension $\mathfrak{N}$ of the structure $\mathfrak{M}(\bar{a}, \bar{b}, \bar{c})$, the prime over the set $\{\bar{a}, \bar{b}, \bar{c}\}$.

For $\bar{\alpha}$ and $\bar{\beta}$ realizations of q in $\mathfrak{N}$ let, $V_{q,\mathfrak{N}}(\bar{\beta}) := \{\bar{\gamma} \in q(N) \mid \exists n \in \mathbb{Z} \quad \mathfrak{N} \vDash s^{(n)}(\bar{\beta}, \bar{\gamma})\}$ be the elements from realizing q in $\mathfrak{N}$ that can be 'reached' from $\bar{\beta}$ in s-steps, and the same for $\bar{\alpha}$. Let us denote

$$(V_{q,\mathfrak{N}}(\bar{\alpha}), V_{q,\mathfrak{N}}(\bar{\beta})) := \{\bar{\gamma} \in q(N) \mid V_{q,\mathfrak{N}}(\bar{\alpha}) < \gamma < V_{q,\mathfrak{N}}(\bar{\beta})\}.$$

And denote $\tilde{a} := (\bar{a}, \bar{b}, \bar{c})$.

Lemma 6.1 [69, P. 7] *For every $\bar{\gamma}_1, \bar{\gamma}_2 \in (V_{p,\mathfrak{N}}(\bar{a}), V_{p,\mathfrak{N}}(\bar{b})) = q(N)$, $tp_{\leq^*}^c(\bar{\gamma}_1|\{\bar{a}, \bar{b}, \bar{c}\}) = tp_{\leq^*}^c(\bar{\gamma}_2|\{\bar{a}, \bar{b}, \bar{c}\})$.*

Proof of Lemma 6.1

Towards a contradiction let there are $\bar{\gamma}_1, \bar{\gamma}_2 \in (V_{p,\mathfrak{N}}(\bar{a}), V_{p,\mathfrak{N}}(\bar{b}))$, and an $\tilde{a}$-definable sentence H for which $\bar{\gamma}_1 \in H(N, \tilde{a}) <^* \bar{\gamma}_2$. To have a convex set let us replace H by $(H(N, \tilde{a})^+)^-$ if necessary.

Given $k, n_1, n_2 < \omega$ for which $n_1 + n_2 < k$ consider

$$S_{k,n_1,n_2}(H)(\bar{x}, \bar{y}, \tilde{a}) := ((\bar{x} <^* \bar{y} \wedge \neg s^k(\bar{x}, \bar{y})) \rightarrow \exists \bar{z}_1, \exists \bar{z}_2 (\bar{x} < \bar{z}_1 <^* \bar{z}_2 <^* \bar{y} \wedge$$
$$\neg s^{n_1}(\bar{x}, \bar{z}_1) \wedge \neg s^{n_2}(\bar{z}_2, \bar{y}) \wedge H(\bar{z}_1, \bar{x}, \bar{y}, \bar{c}) \wedge \neg H(\bar{z}_2, \bar{x}, \bar{y}, \bar{c}) \wedge s(\bar{z}_1, \bar{z}_2, \tilde{a}))).$$

By the theorem of compactness, we can prove the following:

Claim 6.1 [69, P. 8] *There are 2 non-decreasing functions $s_1, s_2: \omega \rightarrow \omega$ which*

are not constant, and for which there exists $m < \omega$, such that for all $k > m$, and all $\bar{\alpha}'$, $\bar{\beta}' \in (\bar{a}, \bar{b})_{p(N)}$, the following holds: $\mathfrak{N} \vDash S_{k,s_1(k),s_2(k)}(H)(\bar{\alpha}', \bar{\beta}')$.

Take $H_\emptyset(\bar{x}, \tilde{a}) := \neg H(\bar{x}, \tilde{a}) \wedge \exists \bar{y}(s(\bar{y}, \bar{x}) \wedge H(\bar{y}, \tilde{a}))$. Then $H_\emptyset(N, \tilde{a}) \cap q(N) \neq \emptyset$ and $H_\emptyset(N, \tilde{a}) \cap q(N) = \{\gamma_\emptyset\}$ for an element $\gamma_\emptyset \in (V_{q,\mathfrak{N}}(\bar{a}), V_{q,\mathfrak{N}}(\bar{b}))$. Then take

$$G_0(\bar{x}, \tilde{a}) := \exists \bar{z}(H(\bar{x}, \bar{a}, \bar{z}, \bar{c}) \wedge H_\emptyset(\bar{z}, \tilde{a}));$$
$$G_1(\bar{x}, \tilde{a}) := \exists \bar{z}(H(\bar{x}, \bar{z}, \bar{b}, \bar{c}) \wedge H_\emptyset(\bar{z}, \tilde{a})).$$

The sets are located in the following way: $G_0(N, \tilde{a}) < V_{q,\mathfrak{N}}(\bar{\gamma}_\emptyset)$, $V_{p,\mathcal{N}}(\bar{a}) < G_0(N, \tilde{a})^+$ and $V_{q,\mathfrak{N}}(\bar{\gamma}_\emptyset) < G_1(N, \tilde{a})^+, G_1(N, \tilde{a}) < V_{p,\mathfrak{N}}(\bar{b})$. We will also use the next notations:

$$H_0(\bar{x}) := \neg G_0(\bar{x}, \tilde{a}) \wedge \exists y(G_0(\bar{y}, \tilde{a}) \wedge s(\bar{y}, \bar{x}));$$
$$H_1(\bar{x}) := \neg G_1(\bar{x}, \tilde{a}) \wedge \exists y(G_1(\bar{y}, \tilde{a}) \wedge s(\bar{y}, \bar{x}));$$

$$G_{00}(\bar{x}, \tilde{a}) := \exists \bar{z}(H(\bar{x}, \bar{a}, \bar{z}, \bar{c}) \wedge H_0(\bar{z}, \tilde{a}));$$
$$G_{01}(\bar{x}, \tilde{a}) := \exists \bar{z}_1, \bar{z}_2(H(\bar{x}, \bar{z}_1, \bar{z}_2, \bar{c}) \wedge H_0(\bar{z}_1, \tilde{a}) \wedge H_\emptyset(\bar{z}_2, \tilde{a}));$$
$$G_{10}(\bar{x}, \tilde{a}) := \exists \bar{z}_1, \bar{z}_2(H(\bar{x}, \bar{z}_1, \bar{z}_2) \wedge H_\emptyset(\bar{z}_1, \tilde{a}) \wedge H_1(\bar{z}_2, \tilde{a}));$$
$$G_{11}(\bar{x}, \tilde{a}) := \exists \bar{z}(H(\bar{x}, \bar{z}, \bar{b}, \bar{c}) \wedge H_1(\bar{z}, \tilde{a})).$$

By using this construction ω times we get a countable number of $\tilde{a}$-definable formulas, $H_\delta, \delta \in 2^{<\omega}$, with the property that for every sequence $\tau \in 2^\omega$, $\tau(n) \in \{0,1\}$ there exists an n-type $q_\tau \in S_n(\{\tilde{a}\})$, extending the next set of $\tilde{a}$-definable n-formulas: $\Gamma_\tau(x) := \{x < H_{\tau,n}(N, \tilde{a}) \mid \tau(n+1) = 0\} \cup \{H_{\tau,n}(x, \tilde{a}) \mid \tau(n+1) = 1\}$.

What is a contradiction to us assuming that the theory T is a small theory.
□ Lemma 6.1

Lemma 6.1 can be used to imply the following.

Lemma 6.2 [69, P. 8] *For all* $\tilde{\delta}_n := \langle \bar{\delta}_1, \ldots, \bar{\delta}_n \rangle$, $\bar{\delta}_i \in (V_{p,\mathfrak{N}}(\bar{a}), V_{p,\mathfrak{N}}(\bar{b}))$, $1 \leq i \leq n$; *with* $V_{q,\mathfrak{N}}(\bar{\delta}_i) < V_{q,\mathfrak{N}}(\bar{\delta}_{i+1})$ $(1 \leq i \leq (n-1))$, *and* $\bar{\gamma} \in N$ *such that* $tp(\bar{\gamma}|\{\bar{a}\bar{b}, \bar{c}\} \cup \tilde{\delta}_n)$ *is isolated we have that:* $\forall \bar{\gamma}_1, \bar{\gamma}_2 \in (V_{p,\Lambda}(\bar{\delta}_i), V_{p,\Lambda}(\bar{\delta}_{i+1}))$,

$$tp^c(\gamma_1|A \cup \bar{\delta}_n \cup \bar{\gamma} \cup \{\bar{\delta}, \bar{\alpha}\}) = tp^c(\gamma_2|A \cup \bar{\delta}_n \cup \bar{\gamma} \cup \{\bar{\delta}, \bar{\alpha}\}).$$

Take $\mathfrak{N}'$ to be an $\aleph_1$-saturated extension of $\mathfrak{M} \cup \{\tilde{a}\}$.

Having an arbitrary sequence $\tau := \langle \tau(1), \tau(2), \ldots, \tau(i), \ldots \rangle_{i<\omega}$ of 0's and 1's, we will apply the a similar construction to the one given in [62, P. 46] to obtain a countable substructure $\mathfrak{M}_\tau \prec \mathfrak{N}'$, such that for every different sequences $\tau_1 \neq \tau_2$, $\mathfrak{M}_{\tau_1} \not\cong \mathfrak{M}_{\tau_2}$. Let us fix such a sequence τ until the end of the proof.

Take $B'_\tau = \{\bar{e}^i_r \mid r \in \mathbb{Q}, i \in \mathbb{N}\} \cup \{\bar{f}^i_n \mid i \in \mathbb{N}, n \in \{0,1\}, \text{ and } \tau(i) = 0\} \cup \{\bar{f}^i_n \mid i \in \mathbb{N}, n \in \{0,1,2\}, \text{and } \tau(i) = 1\} \subseteq q(N')$ with $V_{q,\mathfrak{N}'}(\bar{e}^i_{r_1}) < V_{q,\mathfrak{N}'}(\bar{e}^i_{r_2}) < V_{q,\mathfrak{N}'}(\bar{f}^i_{n_1}) < V_{q,\mathfrak{N}'}(\bar{f}^i_{n_2}) < V_{q,\mathfrak{N}'}(\bar{e}^{i+1}_r)$, where $i \in \mathbb{N}$, $r_1 < r_2 \in \mathbb{Q}$, $r \in Q$, $n_1 < n_2 \in \{0,1,2\}$. The set $B_\tau := B'\underset{\bar{b} \in B'}{\cup} V_{q,\mathfrak{N}'}(\bar{b})$ is countable, so we can enumerate it, $B_\tau = \{\bar{b}_i \mid i < \omega\}$. We will use the notation $\tilde{b}_n := \langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n \rangle$, $n < \omega$. For the constructed model $\mathfrak{M}_\tau$ it will hold that $q(\mathfrak{M}_\tau) = B_\tau$.

Construction of the model $\mathfrak{M}_\tau$.

Step 1. By Λ_1 we will consider the set of all $\tilde{a}$-definable formulas with one free variable, $\Lambda_1 := \{\psi^1_i(x, \tilde{a}) \mid i < \omega\}$. Take $\psi^1_i(x, \tilde{a}) \in \Lambda_1$ with the least i for which $\mathfrak{N}' \vDash \exists x \psi^1_i(x, \tilde{a})$. In respect that T is a small theory, there is an isolated over $\tilde{a}$ formula $\psi^1_{i,1}(x, \tilde{a}) \subseteq \psi^1_i(x, \tilde{a})$ (that is, a subformula), which as well has an isolated subformula over the set $\{\tilde{a}, \bar{b}_1\}$. If we repeat this construction, we will get a finitely satisfiable infinite decreasing sequence of nested formulas $\psi^1_{i,j}(x, \tilde{a}, \tilde{b}_j)$ which are isolated over parameters: $\ldots \subseteq \psi^1_{i,n+1}(N', \tilde{a}, \tilde{b}_{n+1}) \subseteq \psi^1_{i,n}(N', \tilde{a}, \tilde{b}_n) \subseteq \ldots \subseteq \psi^1_i(N', \tilde{a})$. Let us denote by d_1 realization of the constructed chain. Such a realization exists because the structure $\mathfrak{N}'$ is $\aleph_1$-saturated.

Step 2. Let us take a new formula $\psi^1_i(x, \tilde{a}) \in \Lambda_1$ with the least index i and for which there is a witness in $\mathfrak{N}'$, $\mathfrak{N}' \vDash \exists x \psi^1_i(x, \tilde{a})$, and find a realization d_2 analogically, as the realization d_1.

Now let us take b_1 and construct the set of $(\tilde{a} \cup \{\bar{d}_1\} \cup \{\bar{b}_1\})$-definable one-formulas $\Lambda_2 := \{\psi^2_i(x, \tilde{a}, d_1, \bar{b}_1) | i < \omega\}$. Find the formula $\psi^2_i(x, \tilde{a}, d_1, \bar{b}_1) \in \Lambda_2$ which has not been taken previously and has the least index satisfying $\mathfrak{N}' \vDash \exists x \psi^2_i(x, \tilde{a}, \tilde{b}_1, d_1)$, and find d_3 (which exists because $\mathfrak{N}'$ is $\aleph_1$-saturated) of the next infinite chain of isolated over parameters nested formulas $\psi^2_{i,j}(x, \tilde{a}, d_1, \tilde{b}_j)$: $\ldots \subseteq \psi^2_{i,n+1}(x, \tilde{a}, d_1, \tilde{b}_{n+1}) \subseteq \psi^2_{i,n}(x, \tilde{a}, d_1, \tilde{b}_n) \subseteq \ldots \subseteq \psi^2_i(x, \tilde{a}, d_1, \bar{b}_1)$.

At the end of stage k we will have the next sets: for each m, $1 \leq m \leq k$, the sets $D_m := \{d_1, d_2, \ldots, d_{\frac{(m+1)m}{2}}\}$ (we can have $d_i = d_j$ for some indexes i and j with $1 \leq i < j \leq \frac{(m+1)m}{2}$), the set of $\tilde{a}$-definable one-formulas Λ_1, and for each m, $2 \leq m \leq k$, the sets Λ_m of one-formulas definable over the set $(\{\tilde{a}\} \cup D_{m-1} \cup \tilde{b}_{m-1})$.

Step $k + 1$. For every m, $1 \leq m \leq k$, take a formula $\psi^m_{i_m} \in \Lambda_m$ of a smallest index, which was not considered before, and the set of realizations of which in $\mathfrak{N}'$ is not empty. And choose $d_{\frac{(k+1)k}{2}+m}$ realizing descending chains of isolated sub-formulas of formulas $\psi^m_{i_m}$. By Λ_{k+1} denote the set of all one-formulas which are definable over $(\{\tilde{a}\} \cup D_k \cup \tilde{b}_k)$. Next let us chose $d_{\frac{(k+1)k}{2}+k+1}$ analogically to the previous construction. Denote by D_{k+1} the set $\{d_1, d_2, \ldots, d_{\frac{(k+1)k}{2}+k+1}\}$. And denote $M_\tau := \{\tilde{a}\} \cup B_\tau \cup \underset{i<\omega}{\cup} D_i$.

By the statement of the theorem, for given an arbitrary $\bar{\delta} \in q(N') \backslash B$ and any

41

tuple $\tilde{b}_n$ we have that the type $tp(\delta/\tilde{a}, \tilde{b}_n)$ is non-principal. By the way we have chosen $\bar{d}_i := \langle d_1, d_2, \ldots, d_i \rangle_{i<\omega}$, the type $tp(\bar{d}_i/\tilde{a}, \tilde{b}_n, \bar{d}_{i-1})$ is principal, and $tp(\bar{d}_i/\tilde{a}, \tilde{b}_n)$ is as well. Consequently, Lemma 6.2 proves that the type $tp(\bar{\delta}/\tilde{a}, \tilde{b}_n, \bar{d}_i)$ is not principal, and therefore, that $\bar{\delta}$ is not realized in $\mathfrak{M}_\tau$.

The Tarski-Vaught test shows that the structure $\mathfrak{M}_\tau$ that we have obtained is an elementary substructure of the considered structure $\mathfrak{N}'$.

Let us prove that for every two nonequal sequences of 0's and q's τ_1 and τ_2 the structures $\mathfrak{M}_{\tau_1}$ and $\mathfrak{M}_{\tau_2}$ are not isomorphic. To obtain a contradiction let us consider that $\mathfrak{M}_{\tau_1} \overset{\mu}{\cong} \mathfrak{M}_{\tau_2}$. Now take the least index i for which $\tau_1(i) \neq \tau_2(i)$. For convenience we suppose that $i = 1$ and let $0 = \tau_1(1) \neq \tau_2(1) = 1$. We will use the above mentioned construction of B_τ and B'_τ. If we are having an isomorphism, the set of realizations of a type is mapped into the set of realizations of the same type. Also in respect that the function μ is an isomorphism, for all realizations $\bar{c}_1, \bar{c}_2 \in q(M_{\tau_1})$, we have that $\mathfrak{M}_{\tau_1} \vDash \psi(\bar{c}_1, \bar{c}_2)$ implies $\mathfrak{M}_{\tau_2} \vDash \psi(\mu(\bar{c}_1), \mu(\bar{c}_2))$; $\bar{c}_1 \not\in V_{q, \mathfrak{M}_{\tau_1}}(\bar{c}_2)$ implies $\mu(\bar{c}_1) \not\in V_{q, \mathfrak{M}_{\tau_1}}(\mu(\bar{c}_2))$; and from $\bar{c}_1 \in V_{q, \mathfrak{M}_{\tau_1}}(\bar{c}_2)$ it follows that $\mu(\bar{c}_1) \in V_{q, \mathfrak{M}_{\tau_1}}(\mu(\bar{c}_2))$. That is the $V_{q, \mathfrak{M}_{\tau_1}}$-neighborhoods are connected to $V_{q, \mathfrak{M}_{\tau_2}}$-neighborhoods. Moreover, if there exist no $\bar{c}_3 \in q(M_{\tau_1})$ such that $V_{q, \mathfrak{M}_{\tau_1}}(\mu(\bar{c}_1)) <^* \bar{c}_3 <^* V_{q, \mathfrak{M}_{\tau_1}}(\mu(\bar{c}_2))$, meaning that the $V_{q, \mathfrak{M}_{\tau_1}}$-neighborhoods of $\bar{c}_1$ and $\bar{c}_2$ are ordered in a discrete way by means of $<^*$, then neighborhoods of $\mu(\bar{c}_1)$ and $\mu(\bar{c}_2)$ need to be discretely ordered by means of $<^*$ as well. The same is also true for neighborhoods that are ordered in a dense way. Consequently $\mu(V_{q, \mathfrak{M}_{\tau_1}}(\bar{f}_1^1)) = (V_{q, \mathfrak{M}_{\tau_2}}(\bar{f}_1^1)) \; \mu(V_{q, \mathfrak{M}_{\tau_1}}(\bar{f}_2^1)) = (V_{q, \mathfrak{M}_{\tau_2}}(\bar{f}_2^1))$ (having that $\bar{f}_1^1$ and $\bar{f}_2^1$ are located in two first neighborhoods ordered discretely). It is a contradiction since $\mu^{-1}(V_{q, \mathfrak{M}_{\tau_2}}(\bar{f}_3^1))$ has to be in the dense interval of neighborhoods.

The number of different infinite sequences of 0's and 1's is equal to 2^ω, $I(T \cup p(\tilde{a}), \omega) = 2^\omega$. Any structure of T gives maximally ω models of the theory $T \cup tp(\tilde{a})$, and consequently, $I(T, \omega) = 2^\omega$.

$\square$ Theorem 6.2

As an easy corollary the following holds:

Corollary 6.1 *Let we have a countable small theory T with $I(T, \omega) < 2^\omega$. If there exists such a formula $\psi(\bar{x}, \bar{y}, \bar{c})$ which determines a partial order which has a ψ-chain of an infinite length, then this chain is dense.*

Theorem 6.1 is very powerful in studying the number of countable models. The main theorem of the next chapter can be obtained through using the construction in the proof of the Theorem 6.1.

7 MAXIMALITY OF NUMBER OF COUNTABLE MODELS FOR LINEARLY ORDERED THEORIES

Further by $\mathfrak{N}$ we will be considering a countable saturated model of a theory T which is small. We will study linearly ordered theories and suppose that $<$ is an $\emptyset$ −definable linear order relation.

The formulas of the first order will be often written by the relations in definable sets. For instance:

$x < \varphi(N) \equiv \forall y(\varphi(y) \to x < y);$
$x \in (\beta_1, \beta_2) \equiv \beta_1 < x < \beta_2;$
$\varphi(N) \cap \theta(N) \neq \emptyset \equiv N \vDash \exists x(\varphi(x) \wedge \theta(x));$
$\varphi(N) < \theta(N)^+ \equiv N \vDash \forall t(\forall y(\theta(y) \to y < t) \to \forall x(\varphi(x) \to x < t)).$

For a subset $A \subset N$ (which is not necessary definable) we denote:

$$A^+ := \{\gamma \in N | \forall a \in A : N \vDash a < \gamma\}$$
$$A^- := \{\gamma \in N | \forall a \in A : N \vDash \gamma < a\}.$$

Let $A, B \subseteq N$. Then $A \subset B$ means $A \subseteq B$ and $A \neq B$.
The following is a well-known definition:

Definition 7.1 *Let $A \subset B$. The set A is called to be **convex in** the set B, if*

$$\forall x, y \in A(x < y), \forall z \in B(x < z < y \to z \in A).$$

If A is convex in N (that is, it is convex in the universe of the structure), we say that A is **convex**.

For a formula $\varphi(x, \bar{a})$ a **convex closure** of φ is the following formula φ^c:

$$\varphi^c(x, \bar{a}) := \exists y_1, \exists y_2(\varphi(y_1, \bar{a}) \wedge \varphi(y_2, \bar{a}) \wedge (y_1 \leq x \leq y_2)).$$

For a 1-type $p \in S_1(A)$ a **convex closure** of p is a type p^c, such that

$$p^c = \{\varphi^c(x, \bar{a}) | \varphi(x, \bar{a}) \in p\} \text{ [65, P. 6]}.$$

Definition 7.2 [65, P. 6] *Let A and B be subsets of N, $\varphi(x, y)$ be an A-definable 2-formula. The formula $\varphi(x, y)$ is called **B-stable**, if for every element $\alpha \in B$ there are $\gamma_1, \gamma_2 \in B$, $\gamma_1 < \alpha < \gamma_2$ such that $\gamma_1 < \varphi(\alpha, N) < \gamma_2$, and such that $\varphi(\alpha, N) \cap B \neq \emptyset$.*

If $B = \Theta(N)$ and Θ is an A-definable 1-formula, or $B = p(N)$ and $p \in S_1(A)$ is one-type then we say that $\varphi(x, y)$ is **Θ- stable** or **p- stable**.

Definition 7.3 [65, P. 6] *A B-stable two-formula $\varphi(x,y)$ is* **convex to the right on** *B, if*

$$\forall \alpha \in B, \forall \beta (\beta \in \varphi(\alpha, N) \to \alpha \leq \beta \wedge \forall \gamma \in B(\alpha < \gamma < \beta \to \gamma \in \varphi(\alpha, N))).$$

If Θ is an A-definable 1-formula, or $p \in S_1(A)$ is a 1-type such that $B = \Theta(N)$ or $B = p(N)$, then we say that the formula $\varphi(x,y)$ is convex to the right on $\Theta(x)$ or on $p(x)$.

Definition 7.4 [65, P. 7] *We say that a B-stable 2-formula $\varphi(x,y)$* **is convex to the left on** *the set B, if*

$$\forall \alpha \in B, \forall \beta (\beta \in \varphi(\alpha, N) \to \beta \leq \alpha) \wedge \forall \gamma \in B(\beta < \gamma < \alpha \to \gamma \in \varphi(\alpha, N))).$$

If Θ is an A-definable 1-formula, or $p \in S_1(A)$ is a 1-type such that $B = \Theta(N)$ or $B = p(N)$, then we say that the formula $\varphi(x,y)$ is convex to the left on $\Theta(x)$ or on $p(x)$.

The definitions of a convex to the right, convex to the left, and B-stable formulas generalize the notions for weakly o-minimal theories, which were defined in [66] and [67], and introduced in [68]. The other generalization of p-stability were represented in [52, P. 161]. In this work instead of the notion "p-stable" the notion "p-preserving" is used.

Definition 7.5 [65, P. 7] *1) A convex to the right 2-formula $\varphi(x,y)$* **increases on** *B, if $\forall \alpha, \beta \in B$, $(\alpha < \beta \to \varphi(\beta, N)^+ \subseteq \varphi(\alpha, N)^+)$.*
2) A convex to the left 2-formula $\varphi(x,y)$ **decreases on** *B, if $\forall \alpha, \beta \in B$, $(\alpha < \beta \to \varphi(\alpha, N)^- \subseteq \varphi(\beta, N)^-)$.*

We are interested in the case when $\beta \in \varphi(\alpha, N)$.

Definition 7.6 [65, P. 7] *An A-definable increasing on B (decreasing on B) 2-formula $\varphi(x,y)$ is a* **quasi-successor** *on B, if $\forall \alpha \in B$, $\exists \beta \in \varphi(\alpha, N) \cap B$, $B \cap (\varphi(\beta, N) \backslash \varphi(\alpha, N)) \neq \emptyset$.*

As in the previous definitions, If $\Theta(x)$ is an A-definable 1-formula, or $p \in S_1(A)$ is a 1-type such that $B = \Theta(N)$ or $B = p(N)$, then we say that the formula $\varphi(x,y)$ is a quasi-successor on $\Theta(x)$ or on $p(x)$.

If $\varphi(x,y)$ is a quasi-successor formula, we denote:

$$\varphi^0(x,y) := \{x = y\};$$
$$\varphi^n(x,y) := \exists y_1, \dots, \exists y_{n-1}(\varphi(x,y_1) \wedge \varphi(y_1,y_2) \wedge \dots \wedge \varphi(y_{n-1},y));$$
$$\varphi^{-n}(x,y) := \exists x_1, \dots, \exists x_{n-1}(\varphi(x_1,x) \wedge \varphi(x_2,x_1) \wedge \dots \wedge \varphi(y,x_{n-1}) \wedge$$

44

$$\wedge\, (y \leq x) \wedge_{i=1}^{n-1} x_i \leq x).$$

$\varphi^n(x, y)$ is also a quasi-successor on B.

Let $\varphi(x, y)$ be a quasi-successor on B. For $\alpha \in B$ we consider a neighbourhood the formula φ defines by acting on α:

$$V_{B,\varphi}(\alpha) := \{\gamma \in B \mid \exists n \in \mathbb{Z}, \gamma \in \varphi^n(\alpha, N) \cap B\}.$$

The proof of the following theorem can be obtained as a modification of the presented proof of Theorem 6.1.

Theorem 7.1 [65, P. 8] *Let we are given a countable complete theory T be of (an expansion of) a linear order. Let A be a finite subset of N, p be a 1-type over the set A, $\varphi(x, y)$ be an A-definable quasi-successor on $p(x)$. Then the theory T has 2^ω countable models up to an isomorphism.*

Proof of Theorem 7.1 If the theory T is not small, then by the Theorem 2.3.2 it has the maximal number of countable models up to an isomorphism, and then the theorem is proved. Therefore further we will can consider the theory T to be small.

Without loss of generality we can assume that the formula $\varphi(x, y)$ is convex to the right on the type $p(x)$.

Denote by $q(x, y)$ the followng 2-type: $\{x < y\} \cup p(x) \cup p(y) \cup \{y \not\in \varphi^n(x, N) \mid n < \omega\} \cup \{R(y, x) \mid R(x, y)$ is an A-definable convex to the right on p 2-formula with the condition: $\forall n < \omega, \forall \alpha \in p(N), \varphi^n(\alpha, N) \cap p(N) \subset R(\alpha, N)\} \cup \{L(x, y) | L(x, y)$ is an A-definable convex to the leftt on p 2-formula with the condition: $\forall n < \omega$, $\exists \alpha_1, \alpha_2 \in p(N), \alpha_1 < \varphi^n(\alpha_2, N), \alpha_1 \in L(\alpha_2, N)\}$. The consistence of $q(x, y)$ is verified directly.

Let the tuple $< \alpha, \beta >$ realizes $q(x, y)$. Then fix this tuple until the end of the proof of the Theorem 7.1. Denote

$$(V_{p,\varphi}(\alpha), V_{p,\varphi}(\beta))_{p(N)} := \{\gamma \in p(N) | V_{p,\varphi}(\alpha) < \gamma < V_{p,\varphi}(\beta)\}.$$

Lemma 7.1 [65, P. 8] $\forall \gamma_1, \gamma_2 \in (V_{p,\varphi,}(\alpha), V_{p,\varphi}(\beta))_{p(N)},$

$$tp^c(\gamma_1 | A \cup \{\alpha, \beta\}) = tp^c(\gamma_2 | A \cup \{\alpha, \beta\}).$$

Proof of Lemma 7.1 Let that the conclusion of Lemma is not true, that is there are $\gamma_1, \gamma_2 \in (V_{p,\varphi}(\alpha), V_{p,\varphi}(\beta))_{p(N)},$ there exists $(A \cup \{\alpha, \beta\})$-definable formula with $\gamma_1 \in H(N, \alpha, \beta) < \gamma_2$. We can think of $H(N, \alpha, \beta)$ as being convex. If not we can take an $(A \cup \{\alpha, \beta\})$-definable formula which defines the set $(H(N\alpha, \beta)^+)^-$.

By the theorem of compactness, we have the following:

$(*)$ there exists an A-definable formula $\Theta(x)$ of the type p such that for every elements $\alpha', \beta' \in \Theta(N), \alpha' < \beta'$, $V_{\Theta,\varphi}(\alpha') < V_{\Theta,\varphi}(\beta')$ implies there are $\gamma_1, \gamma_2 \in$

$(V_{\Theta,\varphi}(\alpha'), V_{\Theta,\varphi}(\beta'))_{\Theta(N)}$, with $\gamma_1 \in H(N, \alpha', \beta') < \gamma_2$ and $\gamma_2 \in \varphi(\gamma_1, N)$.

For $k, n_1, n_2 < \omega$ such that $n_1 + n_2 < k$ let us have the following notation:
$S_{k,n_1,n_2}(H)(x,y) := (x < y \wedge y \not\in \varphi^k(x, N)) \rightarrow \exists z_1, \exists z_2 (x < z_1 < z_2 < y \wedge$
$z_1 \not\in \varphi^{n_1}(x, N) \wedge y \not\in \varphi^{n_2}(z_2, N) \wedge z_1 \in H(N, x, y) \wedge H(N, x, y) < z_2 \wedge z_2 \in$
$\varphi(z_1, N))$.

Claim 7.1 [65, P. 9] *There are non-decreasing non-constant functions $s_1, s_2 : \omega \rightarrow \omega$ for which there exists $m < \omega$ such that $\forall k > m, \forall \alpha', \beta' \in (\alpha, \beta)_{p(N)}$ we have that $N \vDash S_{k,s_1(k),s_2(k)}(H)(\alpha', \beta')$.*

Proof of Claim 7.1 In case opposite, by theorem of compactness, we obtain the contradiction with the definition of $H(x, y, \alpha, \beta)$.

$\square$

Denote $H_\emptyset(x, \alpha, \beta) := \neg H(x, \alpha, \beta) \wedge \exists y (\varphi(y, x) \wedge H(y, \alpha, \beta))$. The Lemma 7.1 implies that $H_\emptyset(N, \alpha, \beta) \cap p(N) \neq \emptyset$ and $H_\emptyset(N, \alpha, \beta) \cap p(N) \subset V_{p,\varphi}(\gamma_\emptyset)$ for some element $\gamma_\emptyset \in (V_{p,\varphi}(\alpha), V_{p,\varphi}(\beta))_{p(N)}$.

Now denote

$$G_0(x, \alpha, \beta) := \exists z (H(x, \alpha, z) \wedge H_\emptyset(z, \alpha, \beta)),$$
$$G_1(x, \alpha, \beta) := \exists z (H(x, z, \beta) \wedge H_\emptyset(z, \alpha, \beta)).$$

So, by $(*)$ we have that $G_0(N, \alpha, \beta) < V_{p,\varphi}(\gamma_\emptyset)$, $V_{p,\varphi}(\alpha) < G_0(N, \alpha, \beta)^+$ and $V_{p,\varphi}(\gamma_\emptyset) < G_1(N, \alpha, \beta)^+, G_1(N, \alpha, \beta) < V_{p,\varphi}(\beta)$.

Denote

$$H_0(x) := \neg G_0(x, \alpha, \beta) \wedge \exists y (G_0(y, \alpha, \beta) \wedge \varphi(y, x)),$$
$$H_1(x) := \neg G_1(x, \alpha, \beta) \wedge \exists y (G_1(y, \alpha, \beta) \wedge \varphi(y, x)).$$

Then by the Lemma 7.1 we have $H_0(N, \alpha, \beta) \cap p(N) \neq \emptyset$ and $H_0(N, \alpha, \beta) \cap p(N) \subset V_{p,\varphi}(\gamma_0)$ for some $\gamma_0 \in (V_{p,\varphi}(\alpha), V_{p,\varphi}(\gamma_0))_{p(N)}$. And $H_1(N, \alpha, \beta) \cap p(N) \neq \emptyset$ and $H_1(N, \alpha, \beta) \cap p(N) \subset V_{p,\varphi}(\gamma_1)$ for some $\gamma_1 \in (V_{p,\varphi}(\gamma_\emptyset), V_{p,\varphi}(\beta))_{p(N)}$.

Thus we have $\alpha < H_0(N) < H_\emptyset(N) < H_1(N) < \beta$, and

$$V_{p,\varphi}(\alpha) < V_{p,\varphi}(\gamma_0) < V_{p,\varphi}(\gamma_\emptyset) < V_{p,\varphi}(\gamma_1) < V_{p,\varphi}(\beta),$$
$$H_0(N) \subset V_{p,\varphi}(\gamma_0), H_\emptyset(N) \subset V_{p,\varphi}(\gamma_\emptyset), H_1(N) \subset V_{p,\varphi}(\gamma_1).$$

Then denote

$$G_{00}(x, \alpha, \beta) := \exists z (H(x, \alpha, z) \wedge H_0(z, \alpha, \beta)),$$
$$G_{01}(x, \alpha, \beta) := \exists z_1, z_2 (H(x, z_1, z_2) \wedge H_0(z_1, \alpha, \beta) \wedge H_\emptyset(z_2, \alpha, \beta)),$$
$$G_{10}(x, \alpha, \beta) := \exists z_1, z_2 (H(x, z_1, z_2) \wedge H_\emptyset(z_1, \alpha, \beta) \wedge H_1(z_2 \alpha, \beta)),$$

46

$$G_{11}(x,\alpha,\beta) := \exists z(H(x,z,\beta) \wedge H_1(z,\alpha,\beta)).$$

So, we have $G_{00}(N,\alpha,\beta) < V_{p,\varphi}(\gamma_0)$, $V_{p,\varphi}(\alpha) < G_{00}(N,\alpha,\beta)^+$ and $V_{p,\varphi}(\gamma_0) < G_{01}(N,\alpha,\beta)^+$, $G_{01}(N,\alpha,\beta) < V_{p,\varphi}(\gamma_\emptyset)$, $\quad G_{10}(N,\alpha,\beta) < V_{p,\varphi}(\gamma_1)$, $\quad V_{p,\varphi}(\gamma_\emptyset) < G_{10}(N,\alpha,\beta)^+$, and $V_{p,\varphi}(\gamma_\emptyset) < G_{11}(N,\alpha,\beta)^+$, $G_{11}(N,\alpha,\beta) < V_{p,\varphi}(\beta)$.

Then by the Lemma 7.1 we have the following: $H_{00}(N,\alpha,\beta) \cap p(N) \neq \emptyset$ and $H_{00}(N,\alpha,\beta) \cap p(N) \subset V_{p,\varphi}(\gamma_{00})$ for some $\gamma_{00} \in (V_{p,\varphi}(\alpha), V_{p,\varphi}(\gamma_0)) - p(N)$. Also $H_{01}(N,\alpha,\beta) \cap p(N) \neq \emptyset$ and $H_{01}(N,\alpha,\beta) \cap p(N) \subset V_{p,\varphi}(\gamma_{01})$ for some $\gamma_{01} \in (V_{p,\varphi}(\gamma_0), V_{p,\varphi}(\gamma_\emptyset))_{p(N)}$. And $H_{10}(N,\alpha,\beta) \cap p(N) \neq \emptyset$ and $H_{10}(N,\alpha,\beta) \cap p(N) \subset V_{p,\varphi}(\gamma_{10})$ for some $\gamma_{10} \in (V_{p,\varphi}(\gamma_\emptyset), V_{p,\varphi}(\gamma_1))_{p(N)}$ $H_{11}(N,\alpha,\beta) \cap p(N) \neq \emptyset$ and $H_{11}(N,\alpha,\beta) \cap p(N) \subset V_{p,\varphi}(\gamma_{11})$ for some $\gamma_{11} \in (V_{p,\varphi}(\gamma_1), V_{p,\varphi}(\beta))_{p(N)}$.

By applying this method ω times, we get a countable family of A-definable formulas $H_\delta, \delta \in 2^{<\omega}$ such that for every $\tau \in 2^\omega$, $\tau(n) \in \{0,1\}$ there is $p_\tau \in S_1(A)$, one-type over A which extends the following set of A-definable 1-formulas:

$$\Gamma_\tau(x) := \{x < H_{\tau(1),\dots,\tau(n)}(N,\bar{a},\beta) | \tau(n+1) = 0\} \cup$$
$$\{H_{\tau(1),\dots,\tau(n)}(x,\alpha,\beta) | \tau(n+1) = 1\}.$$

This contradicts with the assumption that the theory T is small.

$\square$

Using the proof of Lemma 7.1 we can get the following:

Lemma 7.2 [65, P. 10] *For each $\alpha_1, \dots, \alpha_n \in (V_{p,\varphi}(\alpha), V_{p,\varphi}(\beta))_{p(N)}$ ($\bar{\alpha}_n := < \alpha_1, \dots, \alpha_n >$) with $V_{p,\varphi}(\alpha_i) < V_{p,\varphi}(\alpha_{i+1})$ $(1 \leq i \leq (n-1))$, for each $\bar{\gamma} \in N$ with $tp(\bar{\gamma}|A \cup \bar{\alpha}_n \cup \{\alpha,\beta\})$ being isolated the following holds: $\forall \gamma_1, \gamma_2 \in (V_{p,\varphi,}(\alpha_i), V_{p,\varphi}(\alpha_{i+1}))$,*

$$tp^c(\gamma_1 | A \cup \bar{\alpha}_n \cup \bar{\gamma} \cup \{\alpha,\beta\}) = tp^c(\gamma_2 | A \cup \bar{\alpha}_n \cup \bar{\gamma} \cup \{\alpha,\beta\}).$$

It follows from the Lemma 7.2 that any element $\gamma \in (V_{p,\varphi,}(\alpha_i), V_{p,\varphi}(\alpha_{i+1}))$ has non-isolated one-type over $A \cup \bar{\alpha}_n \cup \bar{\beta} \cup \{\alpha,\beta\}$ because it is irrational.

Let $2^{<\omega}$ be a set of all finite tuples of elements from $\{0,1\}$. Then for every $\eta \in 2^{<\omega}$, $\eta := < \eta(1), \eta(2), \dots, \eta(n) >$ denote $l(\eta) := n$. Let $\eta \neq \pi \in 2^{<\omega}$, then we say that η less than π $(\eta < \pi)$ if either $\eta \subset \pi \wedge \pi(l(\eta)+1) = 1$ or $\exists i \leq min\{l(\eta), l(\pi)\}$, $\forall j < i, \eta(j) = \pi(j) \wedge \eta(i) = 0 \wedge \pi(i) = 1$.

Let $< \alpha_1, \alpha_2, \dots, \alpha_n, \dots \quad >_{n<\omega}$ be an ω-sequence of elements from the set of realizations $p(N)$, such that $V_{p,\varphi}(\alpha) < V_{p,\varphi}(\alpha_i) < V_{p,\varphi}(\alpha_{i+1}) < V_{p,\varphi}(\beta)$ $(1 \leq i \leq \omega)$.

Then for every $\tau \in 2^\omega$ we will construct, by using the Lemma 7.2, a countable structure $M_\tau \prec N$ such that for every $n < \omega, \alpha_n \in M_\tau$, we have that $\tau(n) = 0 \leftrightarrow$ in the set M_τ there is no element from $(V_{p,\varphi}(\alpha_{2n}), V_{p,\varphi}(\alpha_{2n+1}))_{p(N)}$ and $\tau(n) = 1 \leftrightarrow$

for every $\eta \in 2^{<\omega}$, there exists an element $\alpha_{n,\eta} \in (V_{p,\varphi}(\alpha_{2n}), V_{p,\varphi}(\alpha_{2n+1}))_{p(N)} \cap M_\tau$, such that for every two distinct $\eta \neq \pi \in 2^{<\omega}$ if $\eta < \pi$ then $V_{p,\varphi}(\alpha_{n,\eta}) < V_{p,\varphi\varphi}(\alpha_{n,\pi})$.

Construction of the model M_τ.

Let $\tau \in 2^\omega$. We will construct M_τ as a union of an increasing chain of finite sets $M_\tau = \bigcup_{m<\omega} B_m$, $B_{m-1} \subset A_m \subset B_m$ such that $|B_m|, |A_m| < \omega$; $|B_m \backslash A_m| = m^2$; $tp(B_m \backslash A_m | A_m)$ is isolated and for each $i \leq m$ we have some fix enumeration of $F_1(B_i)$, where $F_1(B_i)$ is the set of all B_i-definable 1-formulas.

Step 0.

Denote $B_0 := A$. Fix some enumeration of $F_1(B_0)$.

Step $m + 1$. By the Lemma 7.2 and by using approach in the choice of γ_η in the proof of the Lemma 7.1 we can obtain

$$A_{m+1} := B_m \cup \{\alpha_{i,\eta} | \eta \in 2^{<\omega}, \; l(\eta) \leq m + 1, \; \tau(i) = 1\}.$$

For every $k < m + 1$ denote $B_{m+1,k} := A_{m+1} \cup \{\beta_{m+1,k'} | k' < k\}$.

Define $\beta_{m+1,k}$. Let $\Theta_{k,j}(x)$ be 1-formula from $F_1(B_k)$ such that $\Theta_{m+1,k}(N) \cap B_{m+1,k} = \emptyset$ and j is minimal with this property. Then take $G(x)$ an arbitrary atom from $F_1(B_{m+1,k})$ (that is for every $K(x) \in F_1(B_{m+1,k})$ if $G(N) \cap K(N) \neq \emptyset$ then $G(N) \subseteq K(N)$) with $G(N) \subseteq \Theta_{k,j}(N)$ and arbitrary element from $\beta_{m+1,k} \in G(N)$. The existence of $G(x)$ follows from our assuming that T is small and because $B_{m+1,k}$ is finite.

Then put $B_{m+1} := \bigcup_{k<m+1} B_{m+1,k}$ and fix some enumeration of $F_1(B_{m+1})$.

Let us to verify that M_τ is model. Consider an arbitrary M_τ-definable 1-formula $\Psi(x,\bar\gamma), \bar\gamma \in M_\tau$ such that $N \vDash \exists x \Psi(x, \bar\gamma)$. Then there exists $k < \omega$ such that $\bar\gamma \cap (B_k \backslash B_{k-1}) \neq \emptyset$. Thus for some $m < \omega, k < m$ we have $N \vDash \Psi(\beta_{m,k}, \bar\gamma), \beta \in M_\tau$. What means that $M_\tau \prec N$.

It is clear that if $\tau \neq \tau' \in 2^\omega$, then in language $L^* := L \cup A \cup \{\alpha, \beta\}$, $M_\tau^* \ncong_{L^*} M_{\tau'}^*$.

Every countable structure of the theory T has no more than ω nonisomorphic models in the extended language L^*. Therefore the original theory T has 2^ω countable non-isomorphic models.

□

This finishes the proof of the main result of this section. By the theorem we obtain that if there exists such a formula quasi-successor, then the countable spectrum of a small theory is maximal.

8 VAUGHT'S CONJECTURE FOR WEAKLY O-MINIMAL THEORIES OF CONVEXITY RANK 1

Let us denote by $\mathfrak{L}$ a countable first order language. In this section we will consider $\mathfrak{L}$–models and suppose that the language $\mathfrak{L}$ contains a symbol $<$ which is interpreted as a binary relation of a linear order in these models. An **open interval** in a model $\mathfrak{M}$ is a definable with parameters subset of M which is of the form $I = \{c \in M : M \vDash a < c < b\}$ for some $a, b \in M \cup \{-\infty, \infty\}$ with $a < b$. In a similar way, we can define **closed, half open–half closed**, etc., **intervals** in $\mathfrak{M}$. An arbitrary element $a \in M$ can be represented as the following interval: $[a, a]$. By an **interval** in the model $\mathfrak{M}$ we will ambiguously mean, any of the above mentioned interval types in $\mathfrak{M}$. Recall that a subset A of the universum M is called **convex** if for each elements $a, b \in A$ and $c \in M$ $a < c < b$ implies $c \in A$.

This section studies the concept of a **weak o-minimality** which was firstly investigated by D. Macpherson, D. Marker, and Ch. Steinhorn in their article [70]. By a **weakly o-minimal model**, or a **weakly ordered-minimal structure** we understand a model $\mathfrak{M} = \langle M, =, <, ... \rangle$ which has a linear order relation such that any parametrically definable subset of $\mathfrak{M}$ can be represented in form of a union of a finite number of convex sets in $\mathfrak{M}$. Let us recall that a model $\mathfrak{M}$ is called **o-minimal** (**ordered-minimal**) if every its definable subset can be represented as a union of a finite number of intervals and points in $\mathfrak{M}$. Thus, weak o-minimality is a generalization of the concept of o-minimality.

Let A and B be arbitrary subsets of a linearly ordered model M. Then the expression $A < B$ means that $a < b$ whenever $a \in A$ and $b \in B$. The expression $A < b$ ($b < A$ respectively) means that $A < \{b\}$ ($\{b\} < A$). By A^+ (A^-) we denote the set of elements $b \in M$ satisfying the condition $A < b$ ($b < A$). For an arbitrary type p by $p(M)$ we denote the set of all elements which realize the type p in model $\mathfrak{M}$. Given an n-tuple $\bar{b} = \langle b_1, b_2, ..., b_n \rangle$ by $\bar{b}_i$ we denote the following tuple: $\langle b_1, b_2, ..., b_i \rangle$ for any $1 \leq i \leq n - 1$.

Given a function f on M by $Dom(f)$ we denote the domain of f, and by $Range(f)$ we denote its range.

A theory T is said to be **binary** if every its formula is equivalent to a Boolean combination of formulas with no more than 2 free variables in T.

In the following definitions we consider M to be a weakly ordered-minimal model, $A, B \subseteq M$, M to be $|A|^+$–saturated, and the types p and $q \in S_1(A)$ to be non-algebraic 1-types.

Let us recall the following definition.

Definition 8.1 [45, P. 230] *A type p said to be not **weakly orthogonal** to a type q ($p \not\perp^w q$) if there is an A-definable formula $H(x, y)$, $a \in p(M)$, and realizations b_1, b_2 from $q(M)$ for which that $b_1 \in H(M, a)$ and $b_2 \notin H(M, a)$.*

In other words, p is **weakly orthogonal** to q ($p \perp^w q$) if $p(x) \cup q(y)$ has a

unique extension to a complete 2-type over A.

Lemma 8.1 ([71], Corollary 34 (iii)) *The non-weak orthogonality relation is a relation of equivalence on the set of all one-types $S_1(A)$.*

Definition 8.2 [72] *We say that a type p is **quite orthogonal** to the type q ($p \perp^q q$) if there is no A–definable bijection $f\colon p(M) \to q(M)$. A **quite o-minimal** theory is a weakly o-minimal theory for which the notions of quite ad weak orthogonality coincide for 1-types over arbitrary sets of structures of the given theory.*

It is obvious every ordered-minimal theory is a quite o-minimal theory.

Definition 8.3 [73] *Let we are given T to be a weakly ordered-minimal theory, let $\mathfrak{M}$ be a sufficiently saturated structure of the theory T, and let the one-formula $\varphi(x)$ be arbitrary and M–definable. To define the convexity rank of the given formula $\varphi(x)$ ($RC(\varphi(x))$) we use the following inductive construction:*
1) $RC(\varphi(x)) \geq 1$ if $\varphi(M)$ is infinite.
2) $RC(\varphi(x)) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i, i \in \omega$, such that:
    – For every $i, j \in \omega$ whenever $i \neq j$ holds that $M \vDash \neg E(b_i, b_j)$; and
    – For every $i \in \omega$ $RC(E(x, b_i)) \geq \alpha$ and the set $E(M, b_i)$ is a convex subset of $\varphi(M)$.
3) $RC(\varphi(x)) \geq \mu$ if $RC(\varphi(x)) \geq \alpha$ for any $\alpha < \mu$ (μ is limit).

If for some element α $RC(\varphi(x)) = \alpha$ then we say that the rank $RC(\varphi(x))$ is **defined**. Otherwise (that is if $RC(\varphi(x)) \geq \alpha$ for any α) we say $RC(\varphi(x)) = \infty$.

In a particular case, a theory has the convexity rank 1 if there are no equivalence relations definable parametrically, which has an infinite number of infinite convex classes. It is obvious that any ordered-minimal theory has convexity rank 1.

To give definition of the **convexity rank of a one-type** p let us consider the following infimum:

$$RC(p) := \inf\{RC(\varphi(x)) \mid \varphi(x) \in p\}.$$

We say that T has **exactly κ (less than κ) countable models** if it has κ (less than κ) countable non-isomorphic structures.

As it is known, in [10, P. 146] was solved the Vaught's conjecture for ordered-minimal theories. Recently in [11, P. 129] the Vaught's conjecture for quite o-minimal theories was solved. From the above works it follows that these theories have the same spectrum, namely such a theory has either continuum of countable structures, or exactly $6^l 3^m$ countable structures for non-negative integers l and m.

In the article [57, P. 1] B.S. Baizhanov and A. Alibek have constructed for any ordinal κ with $4 \leq \kappa \leq \omega$ gave examples of weakly o-minimal theories which have

exactly κ countable structures up to an isomorphism. All these examples have the convexity rank 1. The aim of this section is to investigate the Vaught's conjecture for weakly ordered-minimal theories of convexity rank 1, namely, to describe the countable spectrum of these theories (which already differs from the countable spectrum of ordered-minimal theories):

Theorem 8.1 [74] *Given a weakly o-minimal theory T of a countable signature which has a convexity rank 1. Exactly one of the following possibilities holds:*
    1) *T is countably categorical*
    2) *T is Ehrenfeucht, namely T has k countable structures, where $3 \leq k < \omega$*
    3) *T has ω countable structures*
    4) *T has 2^{ω} countable structures.*

In subsection 2 we will show that there are no so-called p-preserving convex to the right (left) sentences, whose sets of solutions are properly contained in the realization set of a non-algebraic 1-type over an empty set, in a weakly ordered-minimal theory of convexity rank 1 having less than 2^{ω} countable structures (Lemma 8.1.3). As a corollary we get that every non-algebraic type $p \in S_1(\emptyset)$ should be simple and binary.

In subsection 3 will be proved orthogonality of a family of non-algebraic 1-types over the empty set which are pairwise weakly orthogonal (Theorem 8.2.1). In subsection 4 binarity of weakly ordered-minimal theories of the rank of convexity 1 which have less than 2^{ω} countable structures will be showed (Theorem 8.3.1). In Section 5 sets of realizations of non-principal 1-types are investigated (Proposition 8.4.2) and the proof of the main result (Theorem 8.1), that is the solution of Vaught's conjecture for weakly o-minimal theories of convexity rank 1 will be given.

Further in this section we will assume that there exists a large saturated structure which is said to be a monster model, for a given complete weakly ordered-minimal theory T. We will be assuming that every structure under consideration (and particularly, every countable structure of the theory T) is an elementary substructure of the so-called monster model, and that each set is a subset of the universum of the monster model.

Finally, let us note that if T is a weakly ordered-minimal theory of a signature L and A is a finite set, then $T(A) := T \cup \{\varphi(\bar{a}) | \varphi(\bar{x})$ is an L-formula, $\bar{a} \in A$, $\mathfrak{M} \vDash \varphi(\bar{a})$ for some $M \vDash T$ with $A \subseteq M\}$, a theory generated from T by adding to the language L constants for all the elements of the set A is weakly ordered-minimal. Also, if the theory T has less than 2^{ω} countable models, then the new theory $T(A)$ has less than 2^{ω} countable models as well. These observations allow us to generalize the results about one-types in $S_1(\emptyset)$ to analogical results on types in $S_1(A)$, for A an arbitrary finite set.

8.1 Behaviour of 2-formulas and binarity of 1-type

For continuity of narration let us recall the following definitions.

Definition 8.1.1 [65, P. 7; 75] Let $\mathfrak{M}$ be a linearly ordered structure, let $A \subseteq M$, M be $|A|^+$-saturated, and $p \in S_1(A)$ be non-algebraic.

1) An A-definable two-formula $F(x, y)$ is called **p-preserving** (**p–stable**) if there are such elements α, λ_1, $\lambda_2 \in p(M)$ that $p(M) \cap [F(M, \alpha)\backslash\{\alpha\}] \neq \emptyset$ and $\lambda_1 < F(M, \alpha) < \lambda_2$.

2) A p-preserving two-formula $F(x, y)$ is called **convex to the right (convex to the left)** if there is a realization $\alpha \in p(M)$ for which the set $p(M) \cap F(M, \alpha)$ is convex, the realization α is the left (the right) endpoint of $F(M, \alpha)$, and $\alpha \in F(M, \alpha)$.

Definition 8.1.2 [76] *A p-preserving convex to the right (to the left) two-formula $F(x, y)$ is said to be an **equivalence-generating** formula, if for each realizations $\alpha, \beta \in p(M)$ with $M \vDash F(\beta, \alpha)$ we have the following:* $M \vDash \forall x \left(x \geq \beta \rightarrow \left(F(x, \alpha) \leftrightarrow F(x, \beta) \right) \right) (M \vDash \forall x (x \leq \beta \rightarrow (F(x, \alpha) \leftrightarrow F(x, \beta)))).$

Lemma 8.1.1 [76, P. 35] *Let we are given a weakly ordered-minimal structure $\mathfrak{M}$, a subset $A \subseteq M$, and a non-algebraic type $p \in S_1(A)$ over A, also let M to be $|A|^+$–saturated. Let $F(x, y)$ be a p-preserving convex to the right (convex to the left) formula which is equivalence-generating. Then*

1) $G(x, y) := F(y, x)$ is a p-preserving convex to the left (convex to the right) formula which is also equivalence-generating.

2) $E(x, y) := F(x, y) \vee F(y, x)$ is an equivalence relation partitioning $p(M)$ into infinitely many infinite convex classes.

Lemma 8.1.2 [76, P. 35] *Let we are given a weakly ordered-minimal theory $\mathfrak{M}$, a subset $A \subseteq M$, let $\mathfrak{M}$ be an $|A|^+$–saturated structure, $p \in S_1(A)$ be a non-algebraic one-type, and $F(x, y)$ be a p-preserving convex to the right (convex to the left) two-formula. If $F(x, y)$ is not an equivalence-generating formula, then there are realizations $\alpha, \beta \in p(M)$ such that $M \vDash F(\beta, \alpha) \wedge \exists x[\neg F(x, \alpha) \wedge F(x, \beta)].$*

Proposition 8.1.1 [11, P. 133] *Let T be a weakly ordered-minimal theory such that $I(T, \omega) < 2^\omega$, let $\mathfrak{M} \vDash T$, A be a finite subset of M, and the one-type $p \in S_1(A)$ be non-algebraic. Then every formula which is p-preserving convex to the right (to the left) is an equivalence-generating formula.*

Proof of Proposition 8.1.1 Suppose that there is a p-preserving convex to the right (to the left) non-equivalence-generating two-formula $F(x, y)$, then, by the Lemma 8.1.2 this formula is a quasi-successor on the type p. Then by Theorem 5.2.2 T has 2^ω countable structures, contradicting the hypotheses of the proposition.

$\square$

Lemma 8.1.3 [74, P. 1193] *Let T be a weakly ordered-minimal theory of convexity rank 1 having less than 2^ω countable structures, let $\mathfrak{M} \vDash T$, A be a finite subset of M, $p \in S_1(A)$ be a non-algebraic one-type. Then there is no p-preserving*

convex to the right (left) formulas.

Proof of Lemma 8.1.3 Let us suppose, towards a contradiction, that there exists a formula $F(x, y)$ which is p-preserving convex to the right. If this formula is equivalence-generating, then by Lemma 8.1.1 we can define a relation of equivalence $E(x, y) := F(x, y) \lor F(y, x)$ which partitions the realization set of p in some structure of T into infinitely many infinite equivalence classes which contradicts that theory T has convexity rank 1. If $F(x, y)$ is not equivalence-generating, it contradicts Proposition 8.1.1.

$\square$

Let we are given a subset $A \subseteq M$, let $p \in S_1(A)$ be non-algebraic, $n \in \omega$. A tuple $\bar{a} = \langle a_1, a_2, \dots, a_n \rangle \in M^n$ is said to be **increasing** if $a_1 < a_2 < \cdots < a_n$.

A type $p(M)$ is called to be **n-indiscernible over A** if for any increasing n-tuples $\bar{a} = \langle a_1, a_2, \dots, a_n \rangle$, $\bar{a}' = \langle a'_1, a'_2, \dots, a'_n \rangle \in [p(M)]^n$, $tp(\bar{a}/A) = tp(\bar{a}'/A)$; the set $p(M)$ is said to be **indiscernible over A** if for any natural $n \in \omega$ $p(M)$ is n-indiscernible over the set A.

Lemma 8.1.4 [11, P. 137] *Let we are given a weakly o-minimal theory T such that $I(T, \omega) < 2^\omega$, let $p \in S_1(\emptyset)$ be non-algebraic type such that $RC(p) = 1$. Then the set $p(M)$ is indiscernible over $\emptyset$.*

A function will be called **non-trivial** if it is neither a projection function nor the identity function.

Definition 8.1.3 [10, P. 151] *A type $p \in S_1(\emptyset)$ is called **simple** if for any $n \in \omega$ whenever $f(x_1, \dots, x_n)$ is non-trivial $\emptyset$-definable, and $a_1, \dots, a_n$ realize the type p, then $f(a_1, \dots, a_n)$ does not realize p.*

Definition 8.1.4 [77] *Let we are given a one-type $p \in S_1(A)$ which is non-algebraic. The type p is **binary over** the set A if for each $n < \omega$ and each increasing tuples $\bar{b} = \langle b_1, \dots, b_n \rangle$, and $\bar{b}' = \langle b'_1, \dots, b'_n \rangle$ from $[p(M)]^n$ with $tp(\langle b_i, b_j \rangle/A) = tp(\langle b'_i, b'_j \rangle/A)$ for every $1 \le i < j \le n$, $tp(\bar{b}/A) = tp(\bar{b}'/A)$. If the type $p \in S_1(\emptyset)$ is non-algebraic it is binary over the empty set, we say simply that the type p is **binary**.*

Lemma 8.1.5 [11, P. 138] *Given weakly ordered-minimal theory T which has less than the maximal number of countable nonisomorphic structures, given a non-algebraic binary one-type $p \in S_1(\emptyset)$. Then the type p is simple.*

Corollary 8.1.1 [74, P. 1194] *Let we are given a weakly o-minimal theory T of convexity rank 1 with $I(T, \omega) < 2^\omega$. Then every non-algebraic type $p \in S_1(\emptyset)$ is simple and binary.*

8.2 Orthogonality

Lemma 8.2.1 [11, P. 139] *Let T be an arbitrary complete theory, $\mathfrak{M} \vDash T$, $A \subseteq M$, $\mathfrak{M}$ be $|A|^+$-saturated, $m, n < \omega$, $\bar{a} = \langle a_1, \ldots, a_m \rangle$, $\bar{a}' = \langle a'_1, \ldots, a'_m \rangle \in M^m$, $\bar{b} = \langle b_1, \ldots, b_n \rangle$, $\bar{b}' = \langle b'_1, \ldots, b'_n \rangle \in M^n$ such that $tp(\bar{b}/A) = tp(\bar{b}'/A)$, $tp(\langle a_i, b_j \rangle/A) = tp(\langle a'_i, b'_j \rangle/A)$ for all $1 \leq i \leq m$, $1 \leq j \leq n$, and $tp(\langle \bar{a}, \bar{b}_{n-1} \rangle/A) = tp(\langle \bar{a}', \bar{b}'_{n-1} \rangle/A)$. Then if $tp(\langle \bar{a}, \bar{b} \rangle/A) \neq tp(\langle \bar{a}', \bar{b}' \rangle/A)$ then there exists $b''_n \in M$ for which $tp(\bar{b}_{n-1}, b_n \rangle/A) = tp(\langle \bar{b}_{n-1}, b''_n \rangle/A)$, $tp(\langle a_i, b_n \rangle/A) = tp(\langle a_i, b''_n \rangle/A)$ for every $1 \leq i \leq m$, and $tp(\langle \bar{a}, \bar{b}_{n-1}, b_n \rangle/A) \neq tp(\langle \bar{a}, \bar{b}_{n-1}, b''_n \rangle/A)$.*

Lemma 8.2.2 [74, P. 1194] *Given a weakly ordered-minimal theory T which has a convexity rank 1 and such that $I(T, \omega) < 2^{\omega}$, let $\mathfrak{M} \vDash T$, let A be a finite subset of the universe of $\mathfrak{M}$, and let p, q be non-algebraic weakly orthogonal types from $S_1(A)$. Then for each realizations $a, a' \in p(M)$, $b_1 < b_2$, $b'_1 < b'_2 \in q(M)$, $tp(\langle a, b_1, b_2 \rangle/A) = tp(\langle a', b'_1, b'_2 \rangle/A)$ holds.*

Proof of Lemma 8.2.2 Suppose the contrary, that there are realizations $a, a' \in p(M)$, $b_1 < b_2, b'_1 < b'_2 \in q(M)$ such that $tp(\langle a, b_1, b_2 \rangle/A) \neq tp(\langle a', b'_1, b'_2 \rangle/A)$. Because p and q are weakly orthogonal types, we have that

$$tp(\langle a, b_1 \rangle/A) = tp(\langle a', b'_1 \rangle/A) = tp(\langle a, b_2 \rangle/A) = tp(\langle a', b'_2 \rangle/A).$$

By the Lemma 8.2.1 there exists a realization $b''_2 \in q(M)$ such that $tp(\langle a, b_2 \rangle/A) = tp(\langle a, b''_2 \rangle/A), b_1 < b''_2$ and $tp(\langle a, b_1, b_2 \rangle/A) \neq tp(\langle a, b_1, b''_2 \rangle/A)$.

Therefore, there is an A-definable formula $R(x, y, z)$ with $M \vDash R(a, b_1, b_2) \wedge \neg R(a, b_1, b''_2)$. By weak ordered-minimality we may assume the set $R(a, b_1, M)$ to be convex.

Denote $q' := tp(b_1/A \cup \{a\})$. Because the types p and q are weakly orthogonal, $q'(M)$ and $q(M)$ are equal sets. Without loss of generality we can consider that $b_2 < b''_2$. Now let us take the following formula:

$$F(x, b_1) := b_1 \leq x \wedge \exists y [R(a, b_1, y) \wedge x \leq y].$$

It is easy to check that the formula $F(x, y)$ is a q'–preserving convex to the right formula. This contradicts Lemma 8.1.3.

$\square$

Following [70, P. 5435], 1.2, we will consider definable functions from M to its Dedekind completion $\overline{M}$, more precisely to sorts of $\overline{M}$ representing infima or suprema.

Let $A, D \subseteq M$, D be infinite, $Z \subseteq \overline{M}$ be an A–definable sort and a function $f: D \to Z$ be A–definable. We call the function f **locally increasing** (**locally decreasing, locally constant**) on the set D if for every element a belonging to D

there exists an interval $J \subseteq D$ which is infinite and contains $\{a\}$ such that the function f strictly increases (strictly decreases, is constant) on the interval J. The function f is said to be **locally monotonic** on the set D if it locally increases or locally decreases on the set D.

Proposition 8.2.1 [78] *Let we are given a weakly ordered-minimal model $\mathfrak{M}$, a subset $A \subseteq M$, and a non-algebraic one-type $p \in S_1(A)$. Then any A-definable function f such that $p(M) \subseteq Dom(f)$ is either locally monotonic or is locally constant on the set $p(M)$.*

Corollary 8.2.1 [74, P. 1195] *Let we are given be a weakly ordered-minimal model $\mathfrak{M}$ of a convexity rank 1, $A \subseteq M$, let $p \in S_1(A)$ be non-algebraic. Then each A-definable function f such that $p(M) \subseteq Dom(f)$ is either strictly monotonic or constant on $p(M)$.*

Take $A \subseteq B \subseteq M$, let B be finite, and let $p_1, p_2, \ldots, p_s \in S_1(A)$ be non-algebraic one-types. A family of 1-types $\{p_1, \ldots, p_s\}$ is called to be **weakly orthogonal over the set B** if all s-tuples $\langle a_1, \ldots, a_s \rangle \in p_1(M) \times \ldots \times p_s(M)$ satisfy the same type over the set B. We say that a family $\{p_1, \ldots, p_s\}$ of one-types is **orthogonal over B** if for every sequence $(n_1, \ldots, n_s) \in \omega^s$, and every increasing tuples $\bar{a}_1, \bar{a}'_1 \in [p_1(M)]^{n_1}, \ldots, \bar{a}_s, \bar{a}'_s \in [p_s(M)]^{n_s}$ for which $tp(\bar{a}_1/B) = tp(\bar{a}'_1/B), \ldots, tp(\bar{a}_s/B) = tp(\bar{a}'_s/B)$, $tp(\langle \bar{a}_1, \ldots, \bar{a}_s \rangle/B) = tp(\langle \bar{a}'_1, \ldots, \bar{a}'_s \rangle/B)$ holds.

Lemma 8.2.3 [74, P. 1195] *Given a weakly ordered-minimal theory T of a convexity rank 1 with $I(T, \omega) < 2^\omega$, given a model $\mathfrak{M} \models T$, a finite subset $A \subseteq M$, and nonalgebraic $p_1, p_2, \ldots, p_s \in S_1(A)$ which are pairwise weakly orthogonal. Then the family $\{p_1, \ldots, p_s\}$ is weakly orthogonal over the set A.*

Proof of Lemma 8.2.3 The proof is done by induction on $s \geq 2$. The step when $s = 2$ is obvious. Now let us suppose that the condition of the lemma is established for sets with s types, and let us prove it for sets with $s + 1$ types, $\{p_1, \ldots, p_s, p_{s+1}\}$. Towards the contradiction let $\{p_1, \ldots, p_{s+1}\}$ a non-weakly orthogonal family over the set A. Then there exist $s + 1$-tuples $\langle a_1, \ldots, a_s, a_{s+1} \rangle, \langle a'_1, \ldots, a'_s, a'_{s+1} \rangle \in p_1(M) \times \ldots \times p_s(M) \times p_{s+1}(M)$ such that

$$tp(\langle a_1, \ldots, a_s, a_{s+1} \rangle/A) \neq tp(\langle a'_1, \ldots, a'_s, a'_{s+1} \rangle/A).$$

Therefore, there is an A-definable formula $\varphi(x_1, \ldots, x_s, x_{s+1})$ with

$$M \models \varphi(a_1, \ldots, a_s, a_{s+1}) \wedge \neg\varphi(a'_1, \ldots, a'_s, a'_{s+1}).$$

Lemma 8.2.1 implies that there is an element $a''_{s+1} \in p_{s+1}(M)$ with $M \models \neg\varphi(a_1, \ldots, a_s, a''_{s+1})$. Denote $M' = \langle M, A, a_1, \ldots, a_{s-2} \rangle$. It is obvious that $Th(\mathfrak{M}')$ is

still a weakly o-minimal theory of convexity rank 1, and it has less than 2^ω countable structures. The induction hypothesis ensures that $p_{s-1}(M)$, $p_s(M)$ and $p_{s+1}(M)$ remain 1-indiscernible in M', that is p_{s-1}, p_s, and p_{s+1} have unique extensions p'_{s-1}, p'_s and p'_{s+1} respectively to 1-types over the union $A \cup \{a_1, \dots, a_{s-2}\}$, and the types p'_{s-1}, p'_s, and p'_{s+1} are pairwise weakly orthogonal.

Let us rename p'_{s-1}, p'_s, and p'_{s+1} by p_1, p_2, and p_3; also rename $\varphi(a_1, \dots, a_{s-2}, x_{s-1}, x_s, x_{s+1})$ by $\varphi(x_1, x_2, x_3)$; and the constants a_{s-1}, a_s, a_{s+1} and a''_{s+1} by a_1, a_2, a_3 and a''_3. Thereby, we have $\mathfrak{M}' \vDash \varphi(a_1, a_2, a_3) \wedge \neg\varphi(a_1, a_2, a''_3)$, where $a_1 \in p_1(M')$, $a_2 \in p_2(M')$, $a_3, a''_3 \in p_3(M')$.

Without loss of generality we can let $a_3 < a''_3$. By weak ordered-minimality we can consider $\varphi(a_1, a_2, M')$ to be a convex set, and that for each $a'_3 \in p_3(M')$ $\neg\varphi(a_1, a_2, a'_3)$ implies $\varphi(a_1, a_2, M') < a'_3$.

Let us take $f_{a_1}(y) := \sup(\varphi(a_1, y, M'))$, $g_{a_2}(x) := \sup(\varphi(x, a_2, M'))$. If $f_{a_1}(a_2) \in M'$ then $f_{a_1}(a'_2) \in M'$ for each $a'_2 \in p_2(M')$ and f_{a_1} is a function that maps $p_2(M')$ to $p_3(M')$. If $f_{a_1}(a_2) \notin M'$ then f_{a_1} is a definable function from $p_2(M')$ to a definable sort Z. Because $p_2(M')$ stays 1-indiscernible over $\{a_1\}$, $f_{a_1}(y)$ does not change its behavior on $p_2(M')$, and therefore by Corollary 8.2.1 it is either strictly increasing, strictly decreasing, or is constant on $p_2(M')$. If $f_{a_1}(y)$ is constant then p_1 and p_2 are not weakly orthogonal.

Similarly, the same reasoning holds for $p_1(M')$ and $g_{a_2}(x)$. Because $p_1(M')$ is 1-indiscernible in M' over $\emptyset$, then if the function $f_{a_1}(y)$ is strictly increases on $p_2(M')$ then the function $f_{a'_1}(y)$ is strictly increases on $p_2(M')$ for each $a'_1 \in p_1(M')$.

Case 1. The function $f_{a_1}(y)$ is strictly increases on the set $p_2(M')$. If $g_{a_2}(x)$ strictly increases on $p_1(M')$, we take $b_1 \in p_1(M')$ with $a_1 < b_1$ and consider the formula $F(x, a_2, a_1, b_1) := \forall z[\varphi(a_1, a_2, z) \to \varphi(b_1, x, z)] \wedge x \leq a_2$.

Let $p'_2 := tp(a_2/\{a_1, b_1\})$. By Lemma 8.2.2 $p'_2(M') = p_2(M')$. And $F(x, y, a_1, b_1)$ is p'_2-preserving convex to left, it is a contradiction with Lemma 8.1.3.

If $g_{a_2}(x)$ is strictly decreasing on $p_1(M')$ then we take $b_1 \in p_1(M')$ such $b_1 < a_1$, and we have $F(x, y, a_1, b_1)$ to be also p'_2-preserving convex to the left.

Case 2. If the function $f_{a_1}(y)$ is strictly decreasing on $p_2(M')$. If $g_{a_2}(x)$ strictly increases on $p_1(M')$ then take $b_1 \in p_1(M')$ for which $a_1 < b_1$ and consider the following formula: $F(x, a_2, a_1, b_1) := \forall z[\varphi(a_1, a_2, z) \to \varphi(b_1, x, z)] \wedge a_2 \leq x$. In this case $F(x, y, a_1, b_1)$ is a p'_2-preserving convex to the right formula, what contradicts with Lemma 8.1.3.

If $g_{a_2}(x)$ strictly decreased on $p_1(M')$ then take $b_1 \in p_1(M')$ for which $b_1 < a_1$, and we get that the function $F(x, y, a_1, b_1)$ is also p'_2-preserving convex to the right.

$\square$

Theorem 8.2.1 [74, P. 1195] *Let a theory T be weakly ordered-minimal of rank of convexity 1, and such that $I(T, \omega) < 2^\omega$, let $p_1, \dots, p_m \in S_1(\emptyset)$ be non-algebraic pairwise weakly orthogonal. Then the family $\{p_1, \dots, p_m\}$ is orthogonal over $\emptyset$.*

Proof of Theorem 8.2.1 We prove the theorem by induction on $m \geq 2$.

Step 2. The proof is done by induction on (n_1, n_2). We show that for each increasing

$$\bar{a} = \langle a_1, a_2, \ldots, a_{n_1} \rangle, \bar{a}' = \langle a'_1, a'_2, \ldots, a'_{n_1} \rangle \in [p_1(M)]^{n_1},$$
$$\bar{b} = \langle b_1, b_2, \ldots, b_{n_2} \rangle, \bar{b}' = \langle b'_1, b'_2, \ldots, b'_{n_2} \rangle \in [p_2(M)]^{n_2}$$

such that $tp(\bar{a}/\emptyset) = tp(\bar{a}'/\emptyset)$, $tp(\bar{b}/\emptyset) = tp(\bar{b}'/\emptyset)$ the following statement holds: $tp(\langle \bar{a}, \bar{b} \rangle / \emptyset) = tp(\langle \bar{a}', \bar{b}' \rangle / \emptyset)$. The case $(1,1)$ is trivial. Suppose that the step 2 is established for every $(k_1, k_2) <_{lex} (n_1, n_2)$ and let us prove it for (n_1, n_2) with $n_1 + n_2 > 2$.

Towards a contradiction let us suppose that $tp(\langle \bar{a}, \bar{b} \rangle / \emptyset) \neq tp(\langle \bar{a}', \bar{b}' \rangle / \emptyset)$. The weak orthogonality of the types p_1 and p_2 $tp(\langle a_i, b_j \rangle / \emptyset) = tp(\langle a'_i, b'_j \rangle / \emptyset)$ for each $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. Then Lemma 8.2.1 implies that there exists $b''_{n_2} \in p_2(M)$ with $tp(\langle \bar{b}_{n_2-1}, b_{n_2} \rangle / \emptyset) = tp(\langle \bar{b}_{n_2-1}, b''_{n_2} \rangle / \emptyset), tp(\langle a_i, b_{n_2} \rangle / \emptyset) = tp(\langle a_i, b''_{n_2} \rangle / \emptyset)$ for each index $1 \leq i \leq n_1$, and that $tp(\langle \bar{a}, \bar{b}_{n_2-1}, b_{n_2} \rangle / \emptyset) \neq tp(\langle \bar{a}, \bar{b}_{n_2-1}, b''_{n_2} \rangle / \emptyset)$. Now let $A := \{\bar{a}_{n_1-1}, \bar{b}_{n_2-2}\}$. By the induction hypothesis we have that $tp(\langle b_{n_2-1}, b_{n_2} \rangle / A) = tp(\langle b_{n_2-1}, b''_{n_2} \rangle / A)$.

Case 1. $tp(b_{n_2-1}/A) = tp(b_{n_2}/A)$. Let $p'_1(x) := tp(a_{n_1}/A)$, $p'_2(y) := tp(b_{n_2-1}/A)$. By the inductional hypothesis $p'_1 \perp^w p'_2$ and then, by Lemma 8.2.2 we have $tp(\langle a_{n_1}, b_{n_2-1}, b_{n_2} \rangle / A) = tp(\langle a_{n_1}, b_{n_2-1}, b''_{n_2} \rangle / A)$ which is impossible.

Case 2. If $tp(b_{n_2-1}/A) \neq tp(b_{n_2}/A)$. Let p'_1 and p'_2 be as in the Case 1, and that $p'_3(z) := tp(b_{n_2}/A)$. By the inductional hypothesis p'_1, p'_2 and p'_3 are pairwise weakly orthogonal. Therefore by Lemma 8.2.3 $tp(\langle a_{n_1}, b_{n_2-1}, b_{n_2} \rangle / A) = tp(\langle a_{n_1}, b_{n_2-1}, b''_{n_2} \rangle / A)$ which also contradicts the assumption.

Step m. Let us suppose that the theorem is proved for sets of k 1-types for each $k \leq m-1$ and let us prove it for sets of m 1-types. By Lemma 8.2.3 holds the case when $n_1 = 1, n_2 = 1, \ldots, n_m = 1$. Suppose that the step m holds for every $(k_1, k_2, \ldots, k_m) <_{lex} (n_1, n_2, \ldots, n_m)$ and prove it for $(n_1, n_2, \ldots, n_m)$. Let us take arbitrary increasing $\bar{a}_{n_1} \in [p_1(M)]^{n_1}$, $\bar{a}_{n_2} \in [p_2(M)]^{n_2}$, $\ldots$, $\bar{a}_{n_{m-2}} \in [p_{m-2}(M)]^{n_{m-2}}$. The inductive hypothesis ensures that p_{m-1} and p_m have unique extensions to p'_{m-1} and p'_m, the types over $\{\bar{a}_{n_1}, \bar{a}_{n_2}, \ldots, \bar{a}_{n_{m-2}}\}$, that is, $p_{m-1}(M) = p'_{m-1}(M), p_m(M) = p'_m(M)$.

Let $\mathfrak{M}' = \langle M, \bar{a}_{n_1}, \bar{a}_{n_2}, \ldots, \bar{a}_{n_{m-2}} \rangle$. The hypothesis also proves that p'_{m-1} and p'_m are weakly orthogonal in M'. By the Step 2 $\{p'_{m-1}, p'_m\}$ is an orthogonal family in M' over the empty set, and consequently, $\{p_{m-1}, p_m\}$ is orthogonal over the set $\{\bar{a}_{n_1}, \bar{a}_{n_2}, \ldots, \bar{a}_{n_{m-2}}\}$ in M. Since $\{\bar{a}_{n_1}, \bar{a}_{n_2}, \ldots, \bar{a}_{n_{m-2}}\}$ have been taken arbitrarily, $\{p_1, \ldots, p_m\}$ is orthogonal over the empty set.

$\square$

8.3 Non-weakly orthogonal 1-types and binarity of the theory

Further we will use the notion of a (p,q)-splitting formula, which was given in [79] for principal non-algebraic one-types. Given a subset $A \subseteq M$, let $p, q \in S_1(A)$ be non-algebraic types with $p \not\perp^w q$. Extending the notion of a (p,q)-splitting formula to the non-principal case, we say that an A-definable formula $\varphi(x,y)$ is a $(p,q)-$ **splitting formula** if there exists $a \in p(M)$ such that $\varphi(a, M) \cap q(M)$ is convex, there exists a realization $b \in q(M)$ such that $\neg\varphi(a, b)$ holds, and for every $b \in q(M)$ with $\neg\varphi(a, b)$ we have that $\varphi(a, M) \cap q(M) < b$,that is, $[\varphi(a, M) \cap q(M)]^- = q(M)^-$. If $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are (p,q)-splitting formulas we say that the formula $\varphi_1(x, y)$ is **less than** the formula $\varphi_2(x, y)$ if there is such a realization $a \in p(M)$ that $\varphi_1(a, M) \cap q(M) \subset \varphi_2(a, M) \cap q(M)$. It is easy to see that if $p, q \in S_1(A)$ are non-algebraic one-types such that $p \not\perp^w q$, then there is a (p,q)-splitting formula, and also the set of all the (p,q)-splitting formulas is ordered linearly. It is also clear that for arbitrary (p,q)-splitting formula $\varphi(x,y)$ the function defined as $f(x) := \sup(\varphi(x, M))$ is not constant on the type $p(M)$.

Lemma 8.3.1 [74, P. 1197] *Let T be a weakly ordered-minimal theory of convexity rank 1 having less than 2^ω countable structures, $M \vDash T$, $A \subseteq M$, A be finite, $p_1, p_2 \in S_1(A)$ be non-algebraic one-types, and let $p_1 \not\perp^w p_2$. Then*
1) If exists a realization $a \in p_1(M)$ with $dcl(A \cup \{a\}) \cap p_2(M)$ being empty, then there is only one (p_1, p_2)-splitting formula.
2) If exists a realization $a \in p_1(M)$ with $dcl(A \cup \{a\}) \cap p_2(M) \neq \emptyset$, then there are exactly two (p_1, p_2)-splitting formulas $\varphi_1(x, y)$ and $\varphi_2(x, y)$ such that $\varphi_1(x, y)$ is less than $\varphi_2(x, y)$, and $|[\varphi_2(a, M) \backslash \varphi_1(a, M)] \cap p_2(M)| = 1$ for any $a \in p_1(M)$.

Proof of Lemma 8.3.1 1) Assume the contradiction: there are at least 2 (p_1, p_2)-splitting formulas $\varphi_1(x, y)$ and $\varphi_2(x, y)$, and suppose for simplicity that $\varphi_1(x, y)$ is less than the formula $\varphi_2(x, y)$. Then it is obvious that the set $[\varphi_2(a, M) \backslash \varphi_1(a, M)] \cap p_2(M)$ is infinite for any $a \in p_1(M)$.

Consider an arbitrary $b \in p_2(M)$ with $M \vDash \varphi_2(a, b) \wedge \neg\varphi_1(a, b)$, and take the next formula: $F(x, b) := b \leq x \wedge \exists z[\varphi_2(z, b) \wedge \neg\varphi_1(z, b) \wedge \forall t((\varphi_2(z, t) \wedge \neg\varphi_1(z, t) \wedge b \leq t) \to x \leq t)]$. It is easy to see that $F(x, y)$ is a p_2-preserving convex to the fight formula which contradicts Lemma 8.1.3.

2) There is an A-definable function $f: p_1(M) \to p_2(M)$. It is each to show that f is a strictly monotonic bijection. Take arbitrary $a \in p_1(M)$. There exists $b \in p_2(M)$ with $f(a) = b$. Take the following formulas: $\varphi_1(a, y) := y < f(a)$, $\varphi_2(a, y) := y \leq f(a)$. It is clear that $\varphi_1(x, y), \varphi_2(x, y)$ are (p_1, p_2)-splitting, and also that $[\varphi_2(a, M) \backslash \varphi_1(a, M)] \cap p_2(M) = \{b\}$.

Analogical to the part 1) it can be shown that f is unique and there exist no other (p_1, p_2)-splitting formulas.

$\square$

Lemma 8.3.2 [74, P. 1198] *Given a weakly o-minimal theory T of convexity rank*

1 which has less than 2^ω models, let $\mathfrak{M} \vDash T$, $A \subseteq M$, A is finite, $p_1, p_2, p_3 \in S_1(A)$ are different non-algebraic types with $p_1 \not\perp^w p_2$, $p_2 \not\perp^w p_3$. Then for all $a, a' \in p_1(M)$, $b, b' \in p_2(M)$, $c, c' \in p_3(M)$ such that $tp(\langle a, b\rangle/A) = tp(\langle a', b'\rangle/A)$, $tp(\langle a, c\rangle/A) = tp(\langle a', c'\rangle/A)$, $tp(\langle b, c\rangle/A) = tp(\langle b', c'\rangle/A)$ we have that $tp(\langle a, b, c\rangle/A) = tp(\langle a', b', c'\rangle/A)$.

Proof of Lemma 8.3.2 Let us assume the contrary. Therefore there are $a, a' \in p_1(M)$, $b, b' \in p_2(M)$, $c, c' \in p_3(M)$ satisfying the condition of the lemma, and $tp(\langle a, b, c\rangle/A) \neq tp(\langle a', b', c'\rangle/A)$. Then by the Lemma 8.2.1 there is a realization $c'' \in p_3(M)$ with $tp(\langle a, c\rangle/A) = tp(\langle a, c''\rangle/A)$, $tp(\langle b, c\rangle/A) = tp(\langle b, c''\rangle/A)$, and $tp(\langle a, b, c\rangle/A) \neq tp(\langle a, b, c''\rangle/A)$. Then, there exists formula $R(x, y, z)$ which is an A-definable and for which $\mathfrak{M} \vDash R(a, b, c) \wedge \neg R(a, b, c'')$.

Now we prove that a, b, and c are pairwise algebraically independent over A. Otherwise if for example $b \in dcl(A \cup \{a\})$, then there should be an A-definable formula $\theta(x, y)$ with $M \vDash \theta(a, b) \wedge \exists! y \theta(a, y)$. Take the following formula:

$$R'(x, z) := \forall y[\theta(x, y) \to R(x, y, z)].$$

Then $\mathfrak{M} \vDash R'(a, c) \wedge \neg R'(a, c'')$, which is a contradiction with $tp(\langle a, c\rangle/A) = tp(\langle a, c''\rangle/A)$.

Without loss of generality suppose that $c < c''$. Then, changing if necessary, by weak ordered-minimality we may think that $R(a, b, M)$ is a convex set, and for any $c' \in p_3(M)$ $\neg R(a, b, c')$ implies $R(a, b, M) \cap p_3(M) < c'$.

Let $\varphi_{13}(x, y)$ be a (p_1, p_3)-splitting formula, $\varphi_{23}(x, y)$ be a (p_2, p_3)-splitting formula. Then either $\varphi_{13}(a, M) \cap p_3(M) \subseteq \varphi_{23}(b, M) \cap p_3(M)$, or $\varphi_{23}(b, M) \cap p_3(M) \subset \varphi_{13}(a, M) \cap p_3(M)$. If $\varphi_{13}(a, M) \cap p_3(M) = \varphi_{23}(b, M) \cap p_3(M)$, then strict monotonicity of the function $\delta_{23}(x) := \sup \varphi_{23}(x, M)$ on $p_2(M)$ implies that $b \in dcl(A \cup \{a\})$ which contradicts our assumption. If $|[\varphi_{23}(b, M)\backslash\varphi_{13}(a, M)] \cap p_3(M)| = 1$, or $|[\varphi_{13}(a, M)\backslash\varphi_{23}(b, M)] \cap p_3(M)| = 1$, then we can also see that $b \in dcl(A \cup \{a\})$. Suppose that $\varphi_{13}(a, M) \cap p_3(M) \subset \varphi_{23}(b, M) \cap p_3(M)$. Then $|[\varphi_{23}(b, M)\backslash\varphi_{13}(a, M)] \cap p_3(M)| > 1$. Because $tp(\langle a, c\rangle/A) = tp(\langle a, c''\rangle/A)$, then either $c, c'' \in \varphi_{13}(a, M)$, or $c, c'' \in \neg\varphi_{13}(a, M)$. Without loss of generality suppose the first. case Let $p'_1 := tp(a/A \cup \{b\})$, $p'_3 := tp(c/A \cup \{b\})$. It is clear that p'_1 is not weakly orthogonal to the type p'_3, and that $R(x, b, z)$, $\varphi_{13}(x, z)$ are (p'_1, p'_3)-splitting formulas, moreover $|[\varphi_{13}(a, M)\backslash R(a, b, M)] \cap p_3(M)| > 1$ which contradicts Lemma 8.3.1. The case when $\varphi_{23}(b, M) \cap p_3(M) \subset \varphi_{13}(a, M) \cap p_3(M)$ can be considered analogically.

□

Lemma 8.3.3 [74, P. 1199] *Given be a weakly ordered-minimal theory T of convexity rank 1 which has less than 2^ω models of countable cardinality, $\mathfrak{M} \vDash T$, $A \subseteq M$, A finite, $p_1, p_2, p_3 \in S_1(A)$ non-algebraic distinct types. Then for each $a, a' \in p_1(M)$, $b, b' \in p_2(M)$, $c, c' \in p_3(M)$ such that $tp(\langle a, b\rangle/A) = tp(\langle a', b'\rangle/A)$, $tp(\langle a, c\rangle/A) = tp(\langle a', c'\rangle/A)$, and $tp(\langle b, c\rangle/A) = tp(\langle b', c'\rangle/A)$ it holds that*

$$tp(\langle a,\ b,\ c\rangle/A) = tp(\langle a',\ b',\ c'\rangle/A).$$

Proof of Lemma 8.3.3 If p_1, p_2, and p_3 were pairwise weakly orthogonal then the proof follows from Lemma 8.3.2. If p_1, p_2, and p_3 are not weakly orthogonal then the proof is implied by Lemma 8.3.2. So we suppose that $p_1 \perp^w p_2$, but $p_2 \not\perp^w p_3$. Then $p_1 \perp^w p_3$, in other case we get $p_1 \not\perp^w p_2$. Suppose that the result of the Lemma 8.3.3 is not true, therefore there exist $a, a' \in p_1(M)$, $b, b' \in p_2(M)$, $c, c' \in p_3(M)$ satisfying the condition of the lemma, and there is an A-definable formula $R(x, y, z)$ such that $\mathfrak{M} \vDash R(a, b, c) \wedge \neg R(a', b', c')$. By Lemma 8.2.1 there is such a realization $c'' \in p_3(M)$ for which $tp(\langle a, c\rangle/A) = tp(\langle a, c''\rangle/A), tp(\langle b, c\rangle/A) = tp(\langle b, c''\rangle/A)$ and $\mathfrak{M} \vDash \neg R(a, b, c'')$.

Similarly to the proof of Lemma 8.3.2 we can see that a, b, and c are pairwise independent over the set A. Without loss of generality let us suppose that $c < c''$. Changing, if it is necessary, by weak ordered-minimality we consider the set $R(a, b, M)$ to be a convex, and that for each realization $c' \in p_3(M)$, $\neg R(a, b, c')$ implies $R(a, b, M) \cap p_3(M) < c'$.

Let $p'_2 := tp(b/A \cup \{a\})$, $p'_3 := tp(c/A \cup \{a\})$. Therefore $p_2(M) = p'_2(M)$, $p_3(M) = p'_3(M)$, then $R(a, y, z)$ is (p'_2, p'_3)-splitting. Since p_2 and p_3 are not weakly orthogonal, there exist a (p_2, p_3)–splitting formula $\varphi_{23}(x, y)$. You can also see that it is a (p'_2, p'_3) –splitting formula as well. Because $tp(\langle b, c\rangle/A) = tp(\langle b, c''\rangle/A)$ we have that either $c, c'' \in \varphi_{23}(b, M)$, or $c, c'' \in \neg\varphi_{23}(b, M)$. Let us suppose the first case. Then $R(a, b, M) \cap p_3(M) \subset \varphi_{23}(b, M) \cap p_3(M)$, also $|[\varphi_{23}(b, M)\backslash R(a, b, M)] \cap p_3(M)| > 1$, what contradicts to Lemma 8.3.1. The case when $p_1 \not\perp^w p_2$, $p_2 \perp^w p_3$ can be shown analogically.

$\square$

Lemma 8.3.4 [74, P. 1199] *Let T be a weakly ordered-minimal theory of convexity rank 1 having less than 2^ω countable models up to an isomorphism, let $\mathfrak{M} \vDash T$, $A \subseteq M$, the set A be finite, $p_1, p_2 \in S_1(A)$ be non-algebraic one-types over A, and $p_1 \not\perp^w p_2$. Then for each realizations $a, a' \in p_1(M)$, $b_1 < b_2, b'_1 < b'_2 \in p_2(M)$ for which $tp(\langle a, b_1\rangle/A) = tp(\langle a', b'_1\rangle/A)$, and $tp(\langle a, b_2\rangle/A) = tp(\langle a', b'_2\rangle/A)$ we have that $tp(\langle a, b_1, b_2\rangle/A) = tp(\langle a',\ b'_1, b'_2\rangle/A)$.*

Proof of Lemma 8.3.4 Towards the contradiction suppose that there exists an A–definable three-formula $R(x, y, z)$ such that $\mathfrak{M} \vDash R(a, b_1, b_2) \wedge \neg R(a', b'_1, b'_2)$ for some realizations $a, a' \in p_1(M)$, $b_1 < b_2, b'_1 < b'_2 \in p_2(M)$ for which the following holds:

$$tp(\langle a, b_1\rangle/A) = tp(\langle a', b'_1\rangle/A), \text{ and } tp(\langle a, b_2\rangle/A) = tp(\langle a', b'_2\rangle/A).$$

By Lemma 8.2.1 there exists such a realization $b''_2 \in p_2(M)$ that $b_1 < b''_2, tp\left(\frac{\langle a, b_2\rangle}{A}\right) = tp\left(\frac{\langle a, b''_2\rangle}{A}\right)$, and $\mathfrak{M} \vDash \neg R(a, b_1, b''_2)$.

By analogy with the proof of Lemma 8.3.2 it can be established that the elements a, b_1 and b_2 are pairwise algebraically independent over A.

60

Let $p'_1(x):= tp(a/A \cup \{b_1\})$, $p'_2(x):= tp(b_1/A \cup \{a\})$, and $p''_2(x):= tp(b_2/A \cup \{a\})$.

Case 1. $p'_2 = p''_2$. Let us suppose that $b_2 < b''_2$. By weak ordered-minimality we may consider $R(a, b_1, M)$ to be convex. Then consider the formula:

$$F(x, b_1):= b_1 \leq x \wedge \exists t[R(a, b_1, t) \wedge x \leq t].$$

We can easily check that $F(x, y)$ is a p'_2-preserving convex to the right formula. Then we have a contradiction with Lemma 8.1.3.

Case 2. $p'_2 \neq p''_2$. Because $p_1 \not\perp^w p_2$, there exists a (p_1, p_2)–splitting formula $\varphi(x, y)$, and because $p'_2 \neq p''_2$, we have $M \vDash \varphi(a, b_1) \wedge \neg\varphi(a, b_2)$.

Because the function defined as $\delta(x):= \sup\varphi(x, M)$ is strictly monotone on the set $p_1(M)$, there is $a_1 \in p_1(M)$ with the condition $\mathfrak{M} \vDash a < a_1 \wedge \varphi(a_1, b_1) \wedge \neg\varphi(a_1, b_2)$.

Now let us consider the functions: $\delta(x):= \sup\varphi(x, M)$, $f_a(y):= \sup R(a, y, M)$, and $g_{b_1}(x) := \sup R(x, b_1, M)$.

Subcase 2a. δ is strictly decreases on $p_1(M)$.

Let us suppose that the function $f_a(y)$ strictly increases on $p'_2(M)$. If the function $g_{b_1}(x)$ strictly increases on $p'_1(M)$ as well, we take $a_1 \in p_1(M)$ for which $M \vDash a < a_1 \wedge \varphi(a_1, b_1) \wedge \neg\varphi(a_1, b_2)$. Then $\varphi(a_1, M) \cap p_2(M) \subset \varphi(a, M) \cap p_2(M)$. Now let us take the next formulas:

$$\Phi_1(y, b_1, a, a_1):= \varphi(a_1, y) \wedge y \leq b_1 \wedge \forall z[R(a, b_1, z) \rightarrow R(a_1, y, z)]$$
$$\Phi_n(y, b_1, a, a_1):= \varphi(a_1, y) \wedge y \leq b_1 \wedge \forall y_1 \forall z[\neg\Phi_{n-1}(y_1, b_1, a, a_1) \wedge$$
$$\varphi(a_1, y_1) \wedge y_1 \leq b_1 \wedge R(a, y_1, z) \rightarrow R(a_1, y, z)], n \geq 2.$$

Then the following holds: $\Phi_1(M, b_1, a, a_1) \subset \Phi_2(M, b_1, a, a_1) \subset \cdots \subset \Phi_n(M, b_1, a, a_1) \subset \cdots$.

Consider the family of formulas: $p'_2(y) \cup \{y < b_1\} \cup \{\neg\Phi_n(y, b_1, a, a_1)|n \in \omega\}$. This set is locally consistent, and because of that there is $q \in S_1(A \cup \{a, a_1, b_1\})$ which extends this set of formulas, and which is non-isolated. Then $T(A \cup \{a, a_1, b_1\})$ has 2^ω countable structures which contradicts the hypotheses of the lemma.

If $g_{b_1}(x)$ is strictly decreasing on $p'_1(M)$ then take $a_1 \in p_1(M)$ such that

$$\mathfrak{M} \vDash a > a_1 \wedge \varphi(a_1, b_1) \wedge \neg\varphi(a_1, b_2).$$

Considering the same formulas $\Phi_n(y, b_1, a, a_1)$, we will obtain a similar contradiction.

Let us suppose that the function $f_a(y)$ is strictly decreases on $p'_2(M)$. If the function $g_{b_1}(x)$ is strictly increasing on $p'_1(M)$ then take $a_1 \in p_1(M)$ such that $M \vDash a < a_1 \wedge \varphi(a_1, b_1) \wedge \neg\varphi(a_1, b_2)$. In this case we replace $y \leq b_1$ in formulas $\Phi_n(y, b_1, a, a_1)$ by $y \geq b_1$, and we also obtain that $\Phi_1(M, b_1, a, a_1) \subset \Phi_2(M, b_1, a, a_1) \subset \cdots \subset \Phi_n(M, b_1, a, a_1) \subset \cdots$ Next we take the next family

formulas: $p'_2(y) \cup \{y > b_1 \wedge \varphi(a_1, y)\} \cup \{\neg\Phi_n(y, b_1, a, a_1) | n \in \omega\}$. It can be also seen that $T(A \cup \{a, a_1, b_1\})$ has 2^ω countable structures contradicting the hypotheses of Lemma 8.3.4.

Subcase 2b. The function δ is strictly increasing on $p_1(M)$.

Without loss of generality suppose that the functions $f_a(y)$ and $g_{b_1}(x)$ are both strictly increasing on $p'_2(M)$ and $p'_1(M)$ respectively (the other cases can be handled in a similar way). Let $a_1 \in p_1(M)$ be an arbitrary element such that $M \vDash a < a_1 \wedge \varphi(a_1, b_1) \wedge \neg\varphi(a_1, b_2)$. Therefore we have the following: $\varphi(a, M) \cap p_2(M) \subset \varphi(a_1, M) \cap p_2(M)$.

Further in the formulas $\Phi_n(y, b_1, a, a_1)$ we replace $\varphi(a_1, y)$ and $\varphi(a_1, y_1)$ by $\varphi(a, y)$ and $\varphi(a, y_1)$ respectively.

Let for any $i \geq 1$

$$B_i(b_1, a, a_1, z) := \exists y_1 [\neg\Phi_i(y_1, b_1, a, a_1) \wedge \varphi(a, y_1) \wedge R(a, y_1, z)].$$

Suppose that we have already proved that

$$\Phi_1(M, b_1, a, a_1) \subset \Phi_2(M, b_1, a, a_1) \subset \cdots \subset \Phi_i(M, b_1, a, a_1).$$

Step i. Consider $\sup B_i(b_1, a, a_1, M)$. If $\sup(B_i(b_1, a, a_1, M)) > \sup(\varphi(a_1, M))$, then we obtain that $\Phi_i(M, b_1, a, a_1) \subset \Phi_{i+1}(M, b_1, a, a_1)$ and we move to step $i + 1$. Suppose that $\sup B_i(b_1, a, a_1, M) \leq \sup\varphi(a_1, M)$. Because $\sup\varphi(a, M) < \sup B_i(b_1, a, a_1, M)$, then due to δ being strictly increasing, there is a realization $a_1^i \in p_1(M)$ located in the interval $a < a_1^i < a_1$, and $\sup\varphi(a_1^i, M) < \sup B_i(b_1, a, a_1, M)$. It is not hard to see that $\Phi_j(M, b_1, a, a_1^i) \subset \Phi_j(M, b_1, a, a_1)$ for every $1 \leq j \leq i$, that is $\inf\Phi_j(M, b_1, a, a_1^i) > \inf\Phi_j(M, b_1, a, a_1)$, and therefore $\sup B_j(b_1, a, a_1^i, M) > \sup B_j(b_1, a, a_1, M)$ for any $1 \leq j \leq$. Thus, we have

$$\Phi_1(M, b_1, a, a_1^i) \subset \Phi_2(M, b_1, a, a_1^i) \subset \cdots \subset \Phi_{i+1}(M, b_1, a, a_1^i).$$

Let us change for simplicity our notation, replace a_1^i by a_1 and move to the step $i + 1$. Thus, for each $n \in \omega$ we can construct a chain of length n: $\Phi_1(M, b_1, a, a_1) \subset \Phi_2(M, b_1, a, a_1) \subset \cdots \subset \Phi_n(M, b_1, a, a_1)$.

Therefore we also obtain that $T(A \cup \{a, a_1, b_1\})$ has 2^ω countable structures, contradicting the statement of the lemma.

$\square$

Lemma 8.3.5 [74, P. 1201] *Let T be a weakly o-minimal theory of convexity rank 1 which has less than 2^ω countable models, $M \vDash T$, $A \subseteq M$, A be finite, $p_1, p_2 \in S_1(A)$ be non-algebraic one-types over A, and p_1 be non-weakly orthogonal to p_2. Then for every $n_1, n_2 < \omega$ and every increasing $\bar{a} = \langle a_1, a_2, \ldots, a_{n_1} \rangle$, $\bar{a}' = \langle a'_1, a'_2, \ldots, a'_{n_1} \rangle \in [p_1(M)]^{n_1}$, $\bar{b} = \langle b_1, b_2, \ldots, b_{n_2} \rangle$, $\bar{b}' = \langle b'_1, b'_2, \ldots, b'_{n_2} \rangle \in [p_2(M)]^{n_2}$ for each i and j such that $1 \leq i \leq n_1$, $1 \leq j \leq n_2$ $tp(\langle a_i, b_j \rangle / A) =$*

$tp(\langle a'_i, b'_j \rangle / A)$ *the following holds:* $tp(\langle \bar{a}, \bar{b} \rangle / A) = tp(\langle \bar{a}', \bar{b}' \rangle / A)$.

Proof of Lemma 8.3.5 The proof of the lemma is done by induction on (n_1, n_2). The step $(1,1)$ is obvious. Now let us suppose that Lemma 8.3.5 is true for all (k_1, k_2) such that $(k_1, k_2) <_{lex} (n_1, n_2)$. Let us prove the lemma for the case (n_1, n_2) with $n_1 + n_2 > 2$. Suppose the contrary: there exist an A–definable formula $R(\bar{x}, \bar{y})$ and increasing tuples $\bar{a}, \bar{a}' \in [p_1(M)]^{n_1}$, $\bar{b}, \bar{b}' \in [p_2(M)]^{n_2}$ satisfying the hypotheses of the lemma, and such that

$$M \vDash R(\bar{a}, \bar{b}) \wedge \neg R(\bar{a}', \bar{b}').$$

Therefore by Lemma 8.2.1 there exists $b''_{n_2} \in p_2(M)$ for which $b_{n_2-1} < b''_{n_2}$, $tp(\langle a_i, b_{n_2} \rangle / A) = tp(\langle a_i, b''_{n_2} \rangle / A), 1 \leq i \leq n_1$, and $M \vDash R(\bar{a}, \bar{b}_{n_2-1}, b_{n_2}) \wedge \neg R(\bar{a}, \bar{b}_{n_2-1}, b''_{n_2})$.

Denote by B the following set, $B := A \cup \{\bar{a}_{n_1-1}, \bar{b}_{n_2-2}\}$. By the inductive hypothesis $tp(\langle b_{n_2-1}, b_{n_2} \rangle / B) = tp(\langle b_{n_2-1}, b''_{n_2} \rangle / B)$, and $tp(\langle a_{n_1}, b_{n_2} \rangle / B) = tp(\langle a_{n_1}, b''_{n_2} \rangle / B)$.

If $tp(b_{n_2-1}/B) = tp(b_{n_2}/B)$ then by Lemma 8.2.1 we have $tp(\langle a_{n_1}, b_{n_2-1}, b_{n_2} \rangle / B) = tp(\langle a_{n_1}, b_{n_2-1}, b''_{n_2} \rangle / B)$, what contradicts to our assumption. If $tp(b_{n_2-1}/B) \neq tp(b_{n_2}/B)$, then by Lemma 8.3.3 we also have $tp(\langle a_{n_1}, b_{n_2-1}, b_{n_2} \rangle / B) = tp(\langle a_{n_1}, b_{n_2-1}, b''_{n_2} \rangle / B)$.

$\square$

Theorem 8.3.1 [74, P. 1202] *Every weakly ordered-minimal theory of convexity rank 1 which has less than 2^ω countable models is a binary structure.*

Proof of Theorem 8.3.1 Let the one-types $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$ be non-algebraic. The proof will be done by induction on $s \geq 2$, we will show that for every $n_1, n_2, \ldots, n_s < \omega$ and every increasing tuples $\bar{a}_{n_1}, \bar{a}'_{n_1} \in [p_1(M)]^{n_1}$, $\bar{a}_{n_2}, \bar{a}'_{n_2} \in [p_2(M)]^{n_2}$, $\ldots$, $\bar{a}_{n_s}, \bar{a}'_{n_s} \in [p_s(M)]^{n_s}$ such that for every i_1, i_2, j, k: $1 \leq i_1 < i_2 \leq s$, $1 \leq j \leq n_{i_1}$, and $1 \leq k \leq n_{i_2}$ $tp(\langle a^j_{n_{i_1}}, a^k_{n_{i_2}} \rangle / \emptyset) = tp(\langle (a^j_{n_{i_1}})', (a^k_{n_{i_2}})' \rangle / \emptyset)$, $p(\langle \bar{a}_{n_1}, \bar{a}_{n_2}, \ldots, \bar{a}_{n_s} \rangle / \emptyset) = tp(\langle \bar{a}'_{n_1}, \bar{a}'_{n_2}, \ldots, \bar{a}'_{n_s} \rangle / \emptyset)$ holds.

Step $s = 2$. If the type p_1 is weakly orthogonal to the type p_2, then Lemma 8.2.3 and Theorem 8.2.1 imply that the set $\{p_1, p_2\}$ is orthogonal over $\emptyset$, that is the previous formula holds. And if $p_1 \not\perp^w p_2$, then it follows from Lemma 8.3.5.

Now let us suppose that the conjecture holds for every $k \leq s - 1$. And we will prove it for s. Consider the case $n_1 = 1, n_2 = 1, \ldots, n_s = 1$. Assume the contrary: there is $\langle a_1, a_2, \ldots, a_s \rangle$, $\langle a'_1, a'_2, \ldots, a'_s \rangle \in p_1(M) \times p_2(M) \times \ldots \times p_s(M)$ with $tp(\langle a_i, a_j \rangle / A) = tp(\langle a'_i, a'_j \rangle / A)$ holding for every $1 \leq i < j \leq s$, and there is an $\emptyset$–definable formula $R(\bar{x})$ with $\mathfrak{M} \vDash R(a_1, a_2, \ldots, a_s) \wedge \neg R(a'_1, a'_2, \ldots, a'_s)$.

Lemma 8.2.1 implies that there exists a realization $a''_s \in p_s(M)$ such that

$tp(\langle a_i, a''_s\rangle/\emptyset) = tp(\langle a_i, a_s\rangle/\emptyset)$ for every $1 \le i \le s - 1$, and $\mathfrak{M} \vDash \neg R(a_1, a_2, ..., a_{s-1}, a''_s)$.

Let $A := \{a_1, a_2, ..., a_{s-3}\}$ and consider an A-type $p'_{s-2} := tp(a_{s-2}/A)$, $p'_{s-1} := tp(a_{s-1}/A)$, $p'_s := tp(a_s/A)$. From the inductive hypothesis we have

$$tp(\langle a_{s-2}, a_s\rangle/A) = tp(\langle a_{s-2}, a''_s\rangle/A) \text{ and } tp(\langle a_{s-1}, a_s\rangle/A) = tp(\langle a_{s-1}, a''_s\rangle/A).$$

Then Lemma 8.3.3 implies $tp(\langle a_{s-2}, a_{s-1}, a_s\rangle/A) = tp(\langle a_{s-2}, a_{s-1}, a''_s\rangle/A)$ which is a contradiction with our assumption. By this, the case $n_1 = 1, n_2 = 1, ..., n_s = 1$ is proved.

Let the assumption is true for every $(k_1, k_2, ..., k_s) <_{lex} (n_1, n_2, ..., n_s)$, and prove it for $(n_1, n_2, ..., n_s)$. Towards the contradiction: there are increasing tuples $\bar{a}_{n_1}, \bar{a}'_{n_1} \in [p_1(M)]^{n_1}, \bar{a}_{n_2}, \bar{a}'_{n_2} \in [p_2(M)]^{n_2}, ..., \bar{a}_{n_s}, \bar{a}'_{n_s} \in [p_s(M)]^{n_s}$ with the properties $tp(\langle a^j_{n_{i_1}}, a^k_{n_{i_2}}\rangle/\emptyset) = tp(\langle (a^j_{n_{i_1}})', (a^k_{n_{i_2}})'\rangle/\emptyset)$ for each i_1, i_2, j, k: $1 \le i_1 < i_2 \le s$, $1 \le j \le n_{i_1}$, $1 \le k \le n_{i_2}$; and there exists an $\emptyset$–definable formula $R(\bar{x}_{n_1}, \bar{x}_{n_2}, ..., \bar{x}_{n_s})$ such that $\mathfrak{M} \vDash R(\bar{a}_{n_1}, \bar{a}_{n_2}, ..., \bar{a}_{n_s}) \wedge \neg R(\bar{a}'_{n_1}, \bar{a}'_{n_2}, ..., \bar{a}'_{n_s})$.

Then the Lemma 8.2.1 implies that there is a realization $(a^{n_s}_{n_s})'' \in p_s(M)$ with $a^{n_s-1}_{n_s} < (a^{n_s}_{n_s})''$, $tp(\langle a^j_{n_i}, a^{n_s}_{n_s}\rangle/A) = tp(\langle a^j_{n_i}, (a^{n_s}_{n_s})''\rangle/A)$ for all $1 \le i \le s - 1$ and $1 \le j \le n_i$, $M \vDash \neg R(\bar{a}_{n_1}, \bar{a}_{n_2}, ..., \bar{a}_{n_{s-1}}, \bar{a}_{n_s-1}, (a^{n_s}_{n_s})'')$.

Denote $B := \{\bar{a}_{n_1}, \bar{a}_{n_2}, ..., \bar{a}_{n_{s-3}}, \bar{a}_{n_{s-2}-1}, \bar{a}_{n_{s-1}-1}, \bar{a}_{n_s-1}\}$, and take the types: $p'_{s-2} := tp(a_{n_{s-2}}/B), p'_{s-1} := tp(a_{n_{s-1}}/B), p'_s := tp(a_{n_s}/B)$.

The inductive hypothesis guarantees that $tp(\langle a_{n_{s-2}}, a_{n_s}\rangle/B) = tp(\langle a_{n_{s-2}}, a''_{n_s}\rangle/B)$, $tp(\langle a_{n_{s-1}}, a_{n_s}\rangle/B) = tp(\langle a_{n_{s-1}}, a''_{n_s}\rangle/B)$.

Then by the Lemma 8.3.3 we have that $tp(\langle a_{n_{s-2}}, a_{n_{s-1}}, a_{n_s}\rangle/B) = tp(\langle a_{n_{s-2}}, a_{n_{s-1}}, a''_{n_s}\rangle/B)$ which contradicts our assumption.

$\square$

8.4 Sets of realizations of non-principal 1-types

Definition 8.4.1 [71, P. 1390] *Let we are given be a weakly ordered-minimal structure $\mathfrak{M}$, a subset $A \subseteq M$, and a non-algebraic type $p \in S_1(A)$. The type p is* **quasirational to the right (quasirational to the left)** *if there exists an A–definable convex one-formula $U_p(x) \in p$ such that for each sufficiently saturated structure $N \succ M$ $U_p(N)^+ = p(N)^+$ $(U_p(N)^- = p(N)^-)$. A non-principal 1-type is called to be* **quasirational** *if it is either quasirational to the right or it is quasirational to the left. A non-quasirational non-principal 1-type is* **irrational**.

It is obvious that a one-type which is both quasirational to the left and quasirational to the right is principal.

Fact 8.4.1 [74, P. 1203] *Let we are given a weakly ordered-minimal theory T, let*

$\mathfrak{M}$ *be a model of* T, $p \in S_1(\emptyset)$ *be quasirational to the right (quasirational to the left) one-type. Then* $\mathfrak{M}$ *does not contain a greatest (least) realization of the type* p.

Proposition 8.4.1 [71, P. 1390] *Given* T *a weakly ordered-minimal theory,* $\mathfrak{M}$ *a model of* T, $A \subseteq M$, $p, q \in S_1(A)$ *non-algebraic types with* $p \not\perp^w q$. *Then:*
    1) p *is irrational if and only if* q *is irrational;*
    2) p *is quasirational if and only if* q *is quasirational;*

In [57, P. 1] weakly ordered-minimal Ehrenfeucht theories of convexity rank 1 which have non-weakly orthogonal quasirational 1-types over $\emptyset$ were constructed. We present these theories in examples 8.4.1 and 8.4.2.

Example 8.4.1 [74, P. 1203] Let $\mathfrak{M} = \langle M; <, P^1, U^2, c_i \rangle_{i \in \omega}$ be a linearly ordered such that the set M is a disjoint union of interpretations of 1-predicates P and $\neg P$, for which $P(M) < \neg P(M)$. We identify each of the interpretations P and $\neg P$ with the set of rational numbers $\mathbb{Q}$, ordered as usual.

The symbol U interprets a binary relation defined as follows: for all $a, b \in M$, $M \vDash U(a, b)$ if and only if $\mathfrak{M} \vDash P(a) \wedge \neg P(b)$ and (viewing a, b as rationals) $\langle \mathbb{R}; <, + \rangle \vDash b < a + \sqrt{2}$.

The constants c_i interpret an infinite strictly increasing sequence on $P(M)$ such that $\lim_{i \to \infty} c_i = \infty_{P(M)}$.

It can be shown that $Th(\mathfrak{M})$ is a weakly ordered-minimal theory of convexity rank 1. Let $p(x) := \{c_i < x | i \in \omega\} \cup \{P(x)\}$, $q(y) := \{\forall t[U(c_i, t) \to t < y] | i \in \omega\} \cup \{\neg P(y)\}$.

It is obvious that $p, q \in S_1(\emptyset)$ are quasirational to the right, and $p \not\perp^w q$.

We state that $Th(\mathfrak{M})$ has exactly 4 countable pairwise non-isomorphic structures: the first case — p and q are not realized; the second case — the sets of realizations of p and q have the order type $(0,1) \cap \mathbb{Q}$ (the saturated case); the other two cases —the realization set of one of p or q has the order type of $[0,1) \cap \mathbb{Q}$, and the realization set of the second — the order type of $(0,1) \cap \mathbb{Q}$.

Example 8.4.2 [74, P. 1204] Let $\mathfrak{M} = \langle M; <, P_1^1, \ldots, P_n^1, U_1^2, \ldots, U_{n-1}^2, c_i \rangle_{i \in \omega}$ be linearly ordered such that M is a disjoint union of interpretations of the predicates $P_1, \ldots, P_n$ with $P_1(M) < P_2(M) < \cdots < P_n(M)$. We identify each of the interpretations P_k, where $1 \le k \le n$, with the set of rational numbers $\mathbb{Q}$, ordered as usual.

The symbols U_j, where $1 \le j \le n - 1$, interpret binary relations defined as follows: for all $a, b \in M$, $M \vDash U_j(a, b)$ if and only if $M \vDash P_1(a) \wedge P_{j+1}(b)$ and $\langle \mathbb{R}; <, + \rangle \vDash b < a + \sqrt{p_j}$, where p_j is the jth prime number.

The constants c_i interpret an infinite strictly increasing sequence on $P_1(M)$ such that $\lim_{i \to \infty} c_i = \infty_{P_1(M)}$.

It can be shown that $Th(\mathfrak{M})$ is a weakly ordered-minimal theory of the convexity rank 1.

Let

$$p_1(x) := \{c_i < x \mid i \in \omega\} \cup \{P_1(x)\},$$
$$p_j(x) := \{\forall t[U_{j-1}(c_i, t) \to t < x] \mid i \in \omega\} \cup \{P_j(x)\}, 2 \le j \le n.$$

It is clear that $p_1, \ldots, p_n \in S_1(\emptyset)$ are quasirational to the right, and $\{p_1, \ldots, p_n\}$ is pairwise non-weakly orthogonal.

We state that $Th(\mathfrak{M})$ has exactly $n+2$ countable pairwise non-isomorphic structures: the first case — $p_1, \ldots, p_n$ are not realized; the second case — the sets of realizations of each of $p_1, \ldots, p_n$ have the same order type $(0,1) \cap \mathbb{Q}$ (the saturated case); the remaining n cases — the realization set of only one of $p_1, \ldots, p_n$ has the order type $[0,1) \cap \mathbb{Q}$, and the sets of realizations of the remaining types — the order type $(0,1) \cap \mathbb{Q}$.

Here we present examples of Ehrenfeucht weakly ordered-minimal theories of rank of convexity 1 which have non-weakly orthogonal irrational 1-types over $\emptyset$.

Example 8.4.3 [74, P. 1204] Let the structure $\mathfrak{M} = \langle M; <, P^1, U^2, c_i, c'_j \rangle_{i,j \in \omega}$ be linearly ordered, and the set M be a disjoint union of interpretations of unary predicates P and $\neg P$ with $P(M) < \neg P(M)$. We identify each of the interpretations P and $\neg P$ with the set of rational numbers $\mathbb{Q}$, ordered as usual.

The symbol U interprets a binary relation defined as follows: for all $a, b \in M$, $M \vDash U(a,b)$ if and only if $M \vDash P(a) \wedge \neg P(b)$ and $\langle \mathbb{R}; <, + \rangle \vDash b < a + \sqrt{3}$.

The constants c_i and c'_j interpret an infinite strictly increasing and an infinite strictly decreasing sequences on $P(M)$ respectively with $\lim_{i \to \infty} c_i = \sqrt{2}_{P(M)} = \lim_{j \to \infty} c'_j$.

Let us construct all the models of the theory T. They are graphically represented on the Picture 1.

It follows from the definition of U, that $P(x)$ and $\neg P(y)$ are mutually dense:

$$M \vDash \forall x_1 \forall x_2 \big[(P(x_1) \wedge P(x_2) \wedge x_1 < x_2) \Leftarrow \exists y \big(\neg P(y) \wedge U(x_2, y) \wedge \neg U(x_1, y) \big) \big],$$
$$\text{and}$$
$$M \vDash \forall y_1 \forall y_2 \big[(\neg P(y_1) \wedge \neg P(y_2) \wedge y_1 < y_2) \Leftarrow \exists x (P(x) \wedge U(x, y_1) \wedge \neg U(x, y_2)) \big].$$

This means that together with density of ordering on $P(M)$ and on $\neg P(M)$, elementary theory $T = Th(\mathfrak{M})$ admits quantifies elimination. Theory T is weakly ordered-minimal of rank of convexity 1 since each 1-formula which has parameters from the set M can be represented as a Boolean combination of convex 1-formulas and there exists no definable non-trivial equivalence relation. Consideration of quantifier-free n-types gives us that the theory T is a small theory.

Any one-formula from the following set determines a principal one-type over an empty set:

$$\{x < c_0, c_{0'} < x \land P(x), \neg P(y) \land U(c_0, y), \neg U(c_{0'}, y) \land \neg P(y)\}, \{c_i < x <$$
$$c_{i+1}, c_{j+1}' < x < c_{j'}, \neg U(c_i, y) \land U(c_{i+1}, y), \neg U(c_{j+1}, y) \land U(c_j, y) | i, j \in \omega\}$$

$M_1 = M$ $P(M_1)$ $\neg P(M_1)$ $c_1 c_2 \cdots$ $p(M_1)$ $\cdots c_2' c_1'$ $q(M_1)$

M_2 $c_1 c_2 \cdots$ $\cdots c_2' c_1'$

M_3 $c_1 c_2 \cdots$ α $\cdots c_2' c_1'$ $U(\alpha, y)$

M_4 $c_1 c_2 \cdots$ $U(x, \beta)$ $\cdots c_2' c_1'$ β

M_5 $c_1 c_2 \cdots$ α $\cdots c_2' c_1'$ $\cdots U(\alpha, y)$

M_6 $c_1 c_2 \cdots$ α $\cdots c_2' c_1'$ $U(\alpha, y) \cdots$

M_7 $c_1 c_2 \cdots$ α_1 α_2 $\cdots c_2' c_1'$ $\cdots U(\alpha_1, y)$ $U(\alpha_2, y) \cdots$

M_8 $c_1 c_2 \cdots U(x, \beta)$ $\cdots c_2' c_1'$ β

M_9 $c_1 c_2 \cdots$ $U(x, \beta) \cdots c_2' c_1'$ β

M_{10} $c_1 c_2 \cdots U(x, \beta_1)$ $U(x, \beta_2) \cdots c_2' c_1'$ β_1 β_2

M_{11} $c_1 c_2 \cdots$ α $U(x, \beta) \cdots c_2' c_1'$ $\cdots U(\alpha, y)$ β

M_{12} $c_1 c_2 \cdots U(x, \beta)$ α $\cdots c_2' c_1'$ β $U(\alpha, y) \cdots$
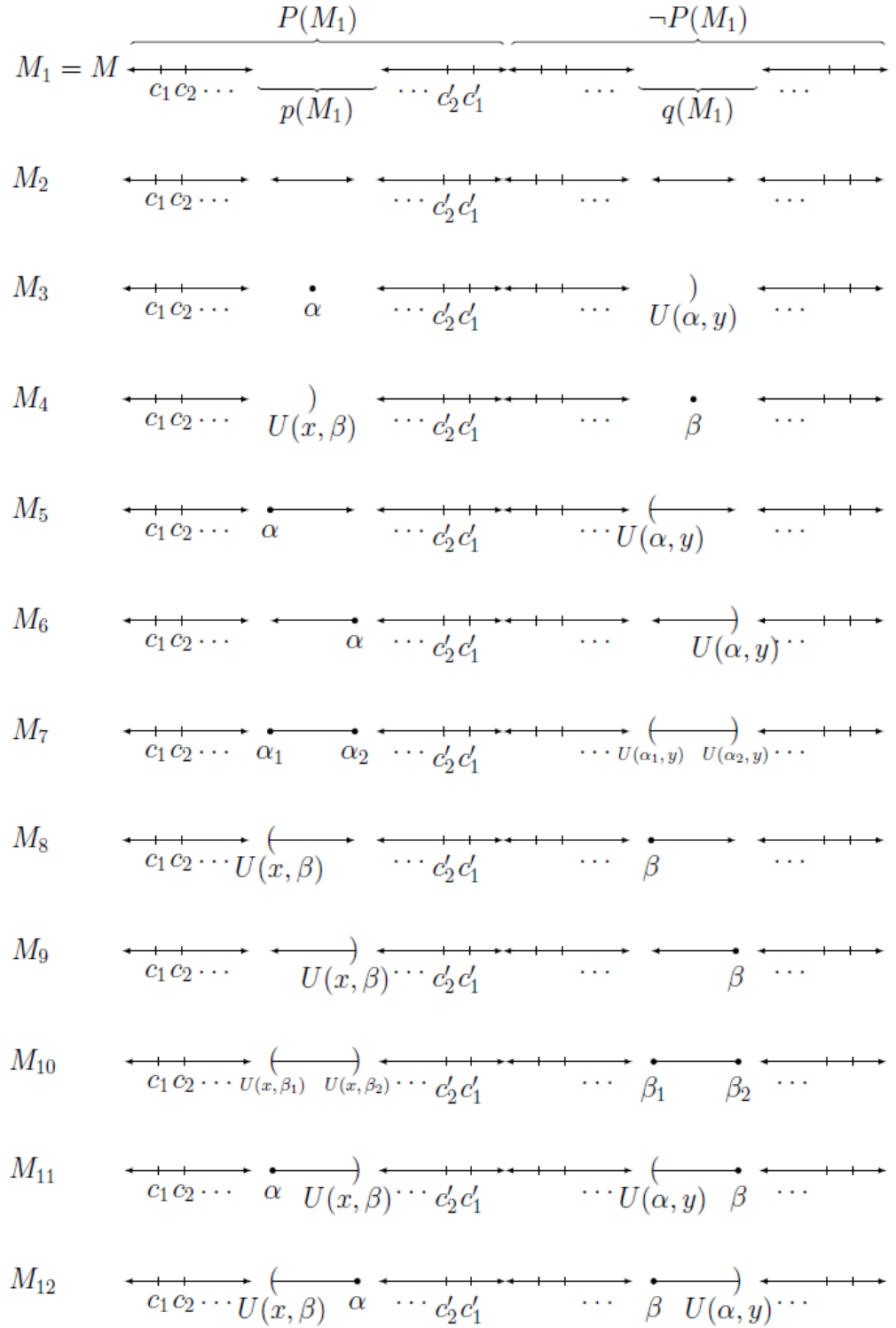
Picture 1

There exist only two non-principal one-types $p(x)$ and $q(y)$ over empty set:

$$p(x) := \{c_i < x < c'_j | i,j \in \omega\} \cup \{P(x)\},$$
$$q(y) := \{\neg U(c_i, y) \land U(c'_j, y) | i,j \in \omega\} \cup \{\neg P(y)\}.$$

Let $\mathfrak{M}_1$ be a countable structure of T such that $p(x)$ and $q(y)$ are omitted in $\mathfrak{M}_1$. Then this structure is isomorphic to the initial model $\mathfrak{M}$ and $\mathfrak{M}_1$ is elementary embedded in any structure of T. So, $\mathfrak{M}_1$ is prime structure of T since it is countable and atomic.

Because T is weakly ordered-minimal, in each structure of the theory T, for each 1-type of T the realization set of this 1-type is a convex set [73]. Denote by $\mathfrak{M}_2$ a countable structure of the theory T such that $P(M_2)$ is non-definable and dense order without end elements. Then it follows from the properties of 2-formula $U(x, y)$ that the set $q(M_2)$ is non-definable and dense ordered without end elements.

Because every countable dense order is embedded into a dense order without end elements and the theory T admits elimination of quantifies any countable structure of T is elementary embedded into M_2. So, $\mathfrak{M}_2$ countable saturated structure of T and therefore, T is small.

Let $\alpha \in p(M_2)$. Because theory T admits quantifies elimination, we have five new one-types over α.

$$p_o(x, \alpha) := p(x) \cup \{x = \alpha\}, \; p_1(x, \alpha) := p(x) \cup \{x < \alpha\},$$
$$p_2(x, \alpha) := p(x) \cup \{\alpha < x\}, \; q_1(y, \alpha) := q(y) \cup \{U(\alpha, y)\},$$
$$q_2(y, \alpha) := q(y) \cup \{\neg U(\alpha, y)\}.$$

One-type p_0 is algebraic, p_1 and p_2 are non-principal, rational one-types over α, q_1 and q_2 are non-principal, quasi-rational, non-rational one-types over α. It follows from definition of $U(x, y)$ that $p_1 =_{RK} q_1, p_2 =_{RK} q_2, p_1 \perp^w q_2, p_2 \perp^w q_1$.

Here, $p_i =_{RK} q_i$ means these two one-types are simultaneously realized or simultaneously omitted in any structure of T. Denote by $\mathfrak{M}_3$ prime structure over α. Then by the previous statement, $p(M_3) = \{\alpha\}$ and $q(M_3) = \emptyset$.

Let $\beta \in q(M_2)$. Because the theory T admits quantifies elimination, we have five new 1-types over β:

$$q'_o(y, \beta) := q(y) \cup \{y = \beta\}, \; q'_1(x, \beta) := q(y) \cup \{y < \beta\},$$
$$q_{2\prime}(y, \beta) := q(y) \cup \{\beta < y\}, \; p_{1\prime}(x, \beta) := p(x) \cup \{U(x, \beta)\},$$
$$p_{2\prime}(x, \beta) := p(x) \cup \{\neg U(x, \beta)\}.$$

The 1-type $q_{0\prime}$ is algebraic, $q_{1\prime}$ and $q_{2\prime}$ are non-principal, rational one-types over β, $p_{1\prime}$ and $p_{2\prime}$ are non-principal, quasi-rational, non-rational one-types over β. It follows from the definition of $U(x, y)$ that $p_{1\prime} =_{RK} q_{1\prime}, p_{2\prime} =_{RK} q_{2\prime}, p_{1\prime} \perp^w q_{2\prime}, p_{2\prime} \perp^w q_1$.

Denote by M_4 prime structure over β. Then by the previous formula, $q(M_4) = \{\beta\}$ and $p(M_4) = \emptyset$. Let $\alpha_1, \alpha_2 \in p(M_2)$, $\beta_1, \beta_2 \in q(M_2)$ such that $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$,

$$M_2 \vDash U(\alpha_1, \beta_1) \wedge \neg U(\alpha_1, \beta_2) \wedge U(\alpha_2, \beta_2).$$

By using properties of types we can construct the countable structures $M_5 - M_{12}$ such that $p(M_i)$ and $q(M_i)$ have different properties on the endpoints of these convex sets.

An arbitrary countable structure $N \vDash T$ is isomorphic to one of these twelve countable structures, represented. Indeed, consider two convex sets $p(N)$ and $q(N)$. It follows from the previous statements that if one of these two sets is singleton then second is empty set. If one of these two sets has more two elements, then both sets are infinite. If one of these two sets has endpoint, say left (right), then left (right) side of second set is definable.

Example 8.4.4 [74, P. 1206] Let $\mathfrak{M} = \langle M; <, P_1^1, \ldots, P_n^1, U_1^2, \ldots, U_{n-1}^2, c_i, c'_j \rangle_{i,j \in \omega}$ be linearly ordered, which universum M is a disjoint union of interpretations of unary predicated $P_1, \ldots, P_n$ with $P_1(M) < P_2(M) < \cdots < P_n(M)$. We identify each of the interpretations $P_1, \ldots, P_n$ with the set of rational numbers $\mathbb{Q}$, ordered as usual.

The symbols U_j, $1 \leq j \leq n-1$, interpret binary relations defined as follows: for all $a, b \in M$, $\mathfrak{M} \vDash U_j(a, b)$ if and only if $M \vDash P_1(a) \wedge \neg P_{j+1}(b)$ and $\langle \mathbb{R}; <, + \rangle \vDash b < a + \sqrt{p_j}$, where p_j is the jth prime number greater than 2.

The constants c_i and c'_j interpret an infinite strictly increasing and an infinite strictly decreasing sequences on $P_1(M)$ respectively, moreover $\lim_{i \to \infty} c_i = \sqrt{2}_{P_1(M)} = \lim_{j \to \infty} c'_j$.

It can be seen that $Th(M)$ is a weakly ordered-minimal theory of rank of convexity equal to 1. Denote

$$p_1(x) := \{c_i < x < c'_j | i, j \in \omega\} \cup \{P_1(x)\},$$
$$p_l(x) := \{\forall t[U_l(c_i, t) \to t < x] | i \in \omega\} \cup \{U_l(c'_j, y) | j \in \omega\} \cup \{P_l(x)\}, 2 \leq l \leq n.$$

It is obvious that $p_1, \ldots, p_n \in S_1(\emptyset)$ are irrational types, and that $\{p_1, \ldots, p_n\}$ is a pairwise non-weakly orthogonal family.

Let $1 \leq j \leq n-1$ and for every $1 \leq 0 \leq n-1$ such that $i \neq j$ denote the following: $U_{ij}(y, z) := \exists t[\neg U_i(t, y) \wedge P_1(t) \wedge U_j(t, z)]$. We state that this formula is (p_{i+1}, p_{j+1})-splitting.

Fact 8.4.5 [74, P. 1206] *The following is true for each $1 \leq i \neq j \leq n-1$:*
1) The set $U_j(a, M)$ $(U_{ij}(a, M))$ has no endpoint from the right in M for all $a \in P_1(M)$ $a \in P_{i+1}(M)$);
2) The set $U_j(M, b)$ $(U_{ij}(M, b))$ has no endpoint from the left in M for all $b \in P_{j+1}(M)$.

Fact 8.4.6 [74, P. 1206] The next is true for each $i \leq i \neq j \leq n-1$:

1) $f_j(x) := sup U_j(x, M)$ *strictly increases on* $P_1(M)$;

2) $f_{ij}(x) := \sup U_{ij}(x, M)$ *strictly increases on* $P_{i+1}(M)$.

We state that theory $Th(\mathfrak{M})$ has exactly $4n + 2 + 2C_n^2$ (where C_n^2 is the binomial coefficient, that is, the theory has exactly $2C_n^2 = n(n-1)$) countable pairwise non-isomorphic structures: the first case — $p_1, \ldots, p_n$ are not realized; the second case — the sets of realizations of each of the types $p_1, \ldots, p_n$ have the order type $(0,1) \cap \mathbb{Q}$ (the saturated case); the following n cases — only one of $p_1, \ldots, p_n$ is omitted by a singleton, and the remaining types are not realized; the following $3n$ cases —the realization set of only one of $p_1, \ldots, p_n$ has the order type $[0,1) \cap \mathbb{Q}$, $(0,1] \cap \mathbb{Q}$ or $[0,1] \cap \mathbb{Q}$, andthe realization set of the remaining types — the order type $(0,1) \cap \mathbb{Q}$; and the remaining $2C_n^2$ cases —the realization set of one of the types $p_1, \ldots, p_n$ has the order type $[0,1) \cap \mathbb{Q}$, and the realization set of another one of $p_1, \ldots, p_n$ has the order type $(0,1] \cap \mathbb{Q}$, and the sets of realizations of the remaining types have the order type $(0,1) \cap \mathbb{Q}$.

Indeed, understand that if there is countable $M' \vDash T$ such that $p_1(M')$ has the order type $[0,1) \cap \mathbb{Q}$ $((0,1] \cap \mathbb{Q})$ and $p_2(M')$ has the order type $(0,1] \cap \mathbb{Q}$ $([0,1) \cap \mathbb{Q})$ then for any $3 \le j \le n$ $p_j(M') \ne \emptyset$ and $p_j(M')$ has no endpoints in M'.

If there exists $3 \le j \le n$ with $p_j(M') = \emptyset$ then taking $a_1, a_2 \in p_1(M')$ with $a_1 < a_2$ we obtain that $f_{j-1}(a_1) = f_{j-1}(a_2)$, which is a contradiction with Fact 0.4.

Suppose now that there exists such a natural number $3 \le j \le n$ that the type $p_j(M')$ has at least one endpoint. For simplicity, let the element c be the left endpoint of $p_j(M')$. Then if a is the left endpoint of the set $p_1(M')$, we have that $U_{j-1}(a, M') < p_j(M')$, that is, c is the right endpoint of $U_{j-1}(a, M')$, which is a contradiction with Fact 8.4.2.

□

Proposition 8.4.2 [74, P. 1207] *Given a weakly ordered-minimal theory T of convexity rank 1 which has less than 2^ω countable models, let $\mathfrak{M}$ be the countable saturated structure of T, $p_1 \in S_1(\emptyset)$ be a non-principal type over an empty set. Then the following conditions are true:*

1) If p_1 is irrational and for each one-type $q \in S_1(\emptyset)$ such that $p_1 \not\perp^w q$ there is an $\emptyset$-definable bijection from $p_1(M)$ to $q(M)$, then for each of the following six possibilities there exists a countable structure M_1 of T in which it is exactly realized: $p_1(M_1) = \emptyset$; $|p_1(M_1)| = 1$; $p_1(M_1)$ is order-isomorphic to $(0,1) \cap \mathbb{Q}$, $[0,1) \cap \mathbb{Q}$, $(0,1] \cap \mathbb{Q}$, or $[0,1] \cap \mathbb{Q}$.

2) If the type p_1 is irrational and there exists a family of types $\lambda = \{p_i \in S_1(\emptyset) | p_1 \not\perp^w p_i, 2 \le i \le n\}$ such that for each types $p', p'' \in \lambda$ there is no $\emptyset$-definable bijection from $p'(M)$ to $p''(M)$, then for each of the following $4n + 2 + 2C_n^2$ possibilities there exists a countable structure M_1 of T in which it is exactly realized: $p_i(M_1) = \emptyset$ for any $1 \le i \le n$; $p_i(M_1)$ is order-isomorphic to the set $(0,1) \cap \mathbb{Q}$ for each $1 \le i \le n$; there is $1 \le i \le n$ with $|p_i(M_1)| = 1$ and the remaining $p_j(M_1)$ $(j \ne i, 1 \le j \le n)$ are empty; there exists $1 \le i \le n$ such that

$p_i(M_1)$ *is order-isomorphic to* $[0,1) \cap \mathbb{Q}$, $(0,1] \cap \mathbb{Q}$ *or* $[0,1] \cap \mathbb{Q}$, *and the remaining* $p_j(M_1)$ *($j \neq i, 1 \leq j \leq n$) are order-isomorphic to* $(0,1) \cap \mathbb{Q}$; *there exist distinct* i, j *with* $1 \leq i, j \leq n$ *such that* $p_i(M_1)$ *is order-isomorphic to* $[0,1) \cap \mathbb{Q}$, $p_j(M_1)$ *is order-isomorphic to either* $(0,1] \cap \mathbb{Q}$ *or* $[0,1) \cap \mathbb{Q}$, *and the remaining types* $p_s(M_1)$ *($s \neq i, s \neq j, 1 \leq s \leq n$) are order-isomorphic to* $(0,1) \cap \mathbb{Q}$, *and conversely.*

3) *If* p_1 *is quasirational to the right (or to the left) and for every one-type* $q \in S_1(\emptyset)$ *for which* $p_1 \not\perp^w q$ *there exists an* $\emptyset$-*definable bijective function from* $p_1(M)$ *to* $q(M)$, *then for each of the following three possibilities there exists a countable structure* M_1 *of* T *in which it is exactly realized:* $p_1(M_1) = \emptyset$; $p_1(M_1)$ *is order-isomorphic to* $(0,1) \cap \mathbb{Q}$; $p_1(M_1)$ *is order-isomorphic to* $[0,1) \cap \mathbb{Q}$ *($(0,1] \cap \mathbb{Q}$).*

4) *If* p_1 *is a quasirational type and there exists a family* $\lambda = \{p_i \in S_1(\emptyset) | p_1 \not\perp^w p_i, 2 \leq i \leq n\}$ *such that for each types* $p', p'' \in \lambda$ *there is no* $\emptyset$-*definable bijection from* $p'(M)$ *to* $p''(M)$, *then for each of the following* $n + 2$ *possibilities there exists a countable structure* M_1 *of the theory* T *in which it is exactly realized:* $p_i(M_1) = \emptyset$ *for any* $1 \leq i \leq n$; $p_i(M_1)$ *is order-isomorphic to the set* $(0,1) \cap \mathbb{Q}$ *for each* $1 \leq i \leq n$; *there is a number* $1 \leq i \leq n$ *such that* $p_i(M_1)$ *is order-isomorphic to either* $[0,1) \cap \mathbb{Q}$ *(p_i is a quasirational to the right type), or it is order-isomorphic to* $(0,1] \cap \mathbb{Q}$ *(when the type* p_i *is a quasirational type to the left), and the remaining* $p_j(M_1)$ *($j \neq i, 1 \leq j \leq n$) are order-isomorphic to the* $(0,1) \cap \mathbb{Q}$.

$\square$

Proposition 8.4.3 [74, P. 1208] *Let* T *be a weakly ordered-minimal theory of convexity rank 1 having less than* 2^ω *countable structures. Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be countable structures of* T *such that for all* $p \in S_1(\emptyset)$ $p(M)$ *is order-isomorphic to* $p(N)$. *Then* M *and* N *are isomorphic.*

Proof of Proposition 8.4.3. Consider $\{m_i \mid i \in \omega\}$ and $\{n_i \mid i \in \omega\}$ to be enumerations of the sets M and N respectively.

Step 0. Take A_0^M, A_0^N denote $dcl(\emptyset)$ in $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Define a partial isomorphism $f_0 : A_0^M \to A_0^N$ by an obvious way.

Step 1. Choose the least index $i \in \omega$ such that $m_i \notin A_0^M$. We have that m_i realizes a type p which is not algebraic over $\emptyset$. Next, consider all $q \in S_1(\emptyset)$ with $q \not\perp^w p$. Let $A_1^M := A_0^M \cup p(M) \cup \{q(M) | q \not\perp^w p\}$. Define A_1^N in a similar way. Because the sets $p(M)$ and $p(N)$ are order-isomorphic, we can extend the partial isomorphism from $p(M)$ into $p(N)$, and correspondingly from $q(M)$ into $q(N)$ for any non-weakly orthogonal type $q \in S_1(\emptyset)$. Let $f_1 : A_1^M \to A_1^N$ be the corresponding partial isomorphism. It is clear that f_1 extends f_0.

Step k. Let us suppose we have already constructed f_{k-1}, A_{k-1}^M and A_{k-1}^N for which $f_{k-1} : A_{k-1}^M \to A_{k-1}^N$ is a partial isomorphism. Find the smallest $i \in \omega$ for which $m_i \notin A_{k-1}^M$. Then $m_i \notin dcl(\emptyset)$ and there is a non-algebraic $p \in S_1(\emptyset)$ such that $m_i \in p(M)$. Clearly p is not realized in A_{k-1}^M, and the same is true of any $q \in S_1(\emptyset)$ which is not weakly orthogonal to p. Let $A_k^M := A_{k-1}^M \cup p(M) \cup \{q(M) | q \not\perp^w p\}$. Similarly, A_{k-1}^N is defined, and a partial isomorphism $f_k : A_k^M \to A_k^N$

extending f_{k-1} is constructed.

Finally, we define the desired isomorphism $f: M \to N$ as $f := \bigcup_{k \in \omega} f_k$.

□

Proof of Theorem 8.1 Let us suppose that T is not a countably categorical theory, and it has less than 2^ω countable structures. Let Λ_1, Λ_2 be maximal pairwise weakly orthogonal families of irrational and quasirational 1-types over $\emptyset$ respectively. It is clear that Λ_1 and Λ_2 are finite. Suppose that, if for example λ_1 was infinite, then we would get that the theory T would have 2^ω countable nonisomorphic models.

Let $\Lambda_1 = \{p_1, p_2, \ldots, p_l\}$, $\Lambda_2 = \{q_1, q_2, \ldots, q_m\}$ for some nonnegative integers $l, m < \omega$. Also let $\Lambda_1^i = \{p_s^i | p_i \not\perp^w p_s^i, s \in \omega\}$, $\Lambda_2^j = \{q_k^j | q_j \not\perp^w q_k^j, k \in \omega\}$ for every integers $1 \le i \le l$, $1 \le j \le m$, and let $|\Lambda_1^i| = \kappa_i$, and $|\Lambda_2^j| = \gamma_j$.

If $\kappa_i = 1$ then we set $n_i = 6$, where n_i is the number of possibilities for pairwise non-isomorphic countable structures of T, because by Proposition 8.4.2 the theory T has exactly 6 countable structures with different order types of the realization set of p_i. If $1 < \kappa_i < \omega$ then $n_i \le 4\kappa_i + 2 + 2C_{\kappa_i}^2$. Obviously, $2C_{\kappa_i}^2 < \kappa_i^2$. If $\kappa_i = \omega$ then $n_i \le \omega$.

Further if $\gamma_j = 1$ then we set $t_j = 3$, where t_j is the number of possibilities for pairwise non-isomorphic countable structures of T, because by the Proposition 8.4.2 T has 3 countable structures with different order types of the realization set of q_j. If $1 < \gamma_j < \omega$ then $t_j \le \gamma_j + 2$. If $\gamma_j = \omega$ then $t_j \le \omega$.

We state that the theory T has no more than $\acute{o}_{i=1}^l n_i * \acute{o}_{j=1}^m t_j$ countable structures, where $\Pi_{i=1}^l n_i = n_1 * n_2 * \ldots * n_l$, $\Pi_{j=1}^m t_j = t_1 * t_2 * \ldots * t_m$ and the symbol $*$ is the operation of multiplication of cardinals. It is clear that the product $\Pi_{i=1}^l n_i * \Pi_{j=1}^m t_j$, is greater than or equal to 3 and less than or equal to ω. This holds because by rules of the cardinal arithmetic for the product of finitely many cardinals, each of the cardinals is not greater than ω, is either equal to ω, or is less than ω.

□

By this we have proved the main theorem of the section, which states that the class of weakly o-minimal theories of convexity rank 1 satisfies the Vaught conjecture.

CONCLUSION

The dissertation considers countable models of small theories and their number up to an isomorphism. Among classes under investigation are the classes of linearly ordered theories, partially ordered theories, and dependent, namely, weakly o-minimal of convexity rank 1, theories.

One of the questions was to find theories, which have the maximal, that is, 2^ω, number of countable non-isomorphic models. The main results in this direction are the following:

1) In a countable complete theory of (an expansion of) a linear order if there exists a subset of a finite cardinality of a model of this theory and a non-principal extremely trivial 1-type over this subset, then the given theory will have the maximal number countable non-isomorphic models.

That is, if there exists a type, such that in a prime model over any finite number of realizations it is realized only by those realizations.

2) If in a countable complete theory if there is a formula which defines a partial order on elements, or on tuples of elements with the condition that for every arbitrary natural number there exists a discrete chain of length not less than this natural number, then the given countable complete theory will have 2^ω countable models up to an isomorphism.

3) If in a countable complete theory of (an expansion of) a linear order there exists a formula quasi-successor on some non-principal 1-type, then this theory has the maximal number of countable non-isomorphic models.

The other question was to find a subclass of dependent theories, for which the Vaught hypothesis holds. The main result on this question is the following:

4) The Vaught conjecture holds for the class of weakly o-minimal theories of the convexity rank 1.

That is, in a countable signature a weakly o-minimal theory of convexity rank 1 is either countably categorical; is an Ehrenfeucht theory, namely it has k countable models, for k between 2 and ω; has ω countable models, or has 2^ω, the maximal number of countable models.

Assessment of the completeness of the aims of the work. All the results are new and are based on our own methods and tools. Conditions guaranteeing maximality of the number of countable models were obtained, as well as a subclass of dependent theories satisfying the Vaught conjecture was found. Therefore, the work objectives were completed.

Suggestions on applications of the obtained results. The results obtained in this area of model theory can be used during the study of countable models of countable small theories and during a search of a proof for the Vaught conjecture. For example, the conditions obtained for maximality of the number of countable models imply that a theory which has ω_1 countable models should not satisfy those conditions. Results obtained on the nature of countable nonisomorphic models of small theories can be applied theories of algebraic structures.

Assessment of scientific level of the work in comparison with the

achievements in the scientific direction. The results obtained in comparison with the best achievements of foreign colleagues do not lose and contribute to the study of countable spectrum of small theories.

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